On the Peculiar Properties of Families of Invariant Manifolds of Conservative Systems

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In our paper, we propose a technique for finding peculiar invariant manifolds of conservative systems having first integrals. To find out peculiar invariant manifolds we use the envelope for the family of first integrals of such systems. In the capacity of peculiar invariant manifolds we consider the invariant manifolds on which several first integrals of the system assume a stationary value.
In the present paper, we shall consider several problems, in which the technique of the envelope has been employed: the problem of motion of a rigid body having one fixed point, the problem of motion of a rigid body in fluid, and the problem related to motion of a system of vortices. *Mathematica* computer algebra system has been used for performing computations. It has enabled us to employ our technique in the problems, manual solving of which is problematic because of bulky computations.
Let us consider Brun's problem well-known in dynamics of a rigid body having one fixed point [1] in the capacity of the first problem.

Brun's Problem

Equations of motion of a rigid body in this case write:

$$\begin{aligned}
A\dot{p} &= (B-C)qr - \mu(B-C)\gamma_2\gamma_3, \quad \dot{\gamma}_1 = r\gamma_2 - q\gamma_3,\\
B\dot{q} &= (C-A)rp - \mu(C-A)\gamma_3\gamma_1, \quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1,\\
C\dot{r} &= (A-B)qp - \mu(A-B)\gamma_1\gamma_2, \quad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2.
\end{aligned} \tag{1}$$

Here A, B, and C are the moments of body inertia; p, q, and r are projections of the body angular rate onto the axes bound up with the body; γ_1, γ_2, and γ_3 are directional cosines of angles between the vertical and the axes bound up with the body.

System (1) possesses the family of first integrals:

$$\begin{aligned}
2K &= Ap^2 + Bq^2 + Cr^2 + \mu(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - 2\lambda_1(Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3)\\
&-\lambda_2[A^2p^2 + B^2q^2 + C^2r^2 - \mu(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)] - \lambda_3(\gamma_1^2 + \gamma_2^2 + \gamma_3^2).
\end{aligned} \tag{2}$$

Under the following conditions imposed on λ_1, λ_3:

$$\lambda_1^2 = \mu[(1-A\lambda_2)(1-B\lambda_2)(1-C\lambda_2)], \;\; \lambda_3 = \mu[\lambda_2(AB+BC+CA) - \lambda_2^2 ABC] \tag{3}$$

the integral K assumes a stationary value on the family of invariant manifolds:

$$A'p - \lambda_1\gamma_1 = 0, \quad B'q - \lambda_1\gamma_2 = 0, \quad C'r - \lambda_1\gamma_3 = 0. \tag{4}$$

This follows from the conditions of stationarity for K. These conditions have the form:

$$\begin{aligned}
\partial K/\partial p &= A(A'p - \lambda_1\gamma_1) = 0, \;\; \partial K/\partial\gamma_1 = [\mu(A + \lambda_2 BC) - \lambda_3]\gamma_1 - \lambda_1 Ap = 0,\\
\partial K/\partial q &= B(B'q - \lambda_1\gamma_2) = 0, \;\; \partial K/\partial\gamma_2 = [\mu(B + \lambda_2 CA) - \lambda_3]\gamma_2 - \lambda_1 Bq = 0,\\
\partial K/\partial r &= C(C'q - \lambda_1\gamma_3) = 0, \;\; \partial K/\partial\gamma_3 = [\mu(C + \lambda_2 AB) - \lambda_3]\gamma_3 - \lambda_1 Cr = 0.
\end{aligned} \tag{5}$$

After substituting (3) into (5), the latter writes:

$$A'\frac{\partial K}{\partial\gamma_1} = A\lambda_1(\lambda_1\gamma_1 - A'p), \;\; B'\frac{\partial K}{\partial\gamma_2} = B\lambda_1(\lambda_1\gamma_2 - B'q), \;\; C'\frac{\partial K}{\partial\gamma_3} = C\lambda_1(\lambda_1\gamma_3 - C'r).$$

Here $A' = (1 - A\lambda_2)$, $B' = (1 - B\lambda_2)$, and $C' = (1 - C\lambda_2)$.

Finding Peculiar Properties of Invariant Manifolds

Let us investigate peculiar properties of the family of invariant manifolds (4) with the use of the envelope for the family of first integrals (2). We shall employ a standard procedure of constructing the envelope for the one-parameter family of integrals.

Compute the derivative of K with respect to parameter λ_2 and equate it to zero:

$$\frac{\partial K}{\partial \lambda_2} = -\frac{d\lambda_1}{d\lambda_2}V_1 - \frac{1}{2}V_2 - \frac{1}{2}\frac{d\lambda_3}{d\lambda_2}V_3 = 0.$$

Here

$V_1 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3,\ V_2 = A^2p^2 + B^2q^2 + C^2r^2 - \mu(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2),$
$V_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2.$

After simple transformations, we obtain the following equation:

$$\frac{1}{2}V_2 + \frac{1}{2}\mu(M - 2L\lambda_2)V_3 - \frac{\mu(3L\lambda_2^2 - 2M\lambda_2 + N)}{2\sqrt{\mu(1 - A\lambda_2)(1 - B\lambda_2)(1 - C\lambda_2)}}V_1 = 0, \quad (6)$$

where $N = A + B + C, M = AB + BC + CA, L = ABC$.

Equality (6) turns zero, for example, under the following values of λ_2:

$i)\ \ \lambda_2 = 0$ when $(V_2 + \mu M)^2 - \mu N^2 V_1^2 = 0,\ V_3 = 1;$

$ii)\ \ \lambda_2 = \dfrac{M}{2L}$ when $V_1 = 0, V_2 = 0,\ V_3 = 1;$

$iii)\ \ \lambda_2 = \dfrac{M - \sqrt{M^2 - 3LN}}{3L}$ when $3V_2 + \mu(M + 2\sqrt{M^2 - 3LN})V_3 = 0.$ (7)

Substitute the values of λ_2 (7) into the equations of invariant manifolds under investigation.

These equations will be reduced to the following ones:

$$i)\quad p=\sqrt{\mu}\gamma_1,\ q=\sqrt{\mu}\gamma_2,\ r=\sqrt{\mu}\gamma_3; \tag{8}$$

$$ii)\quad p=-\frac{\sqrt{\mu(A^2(B-C)^2-B^2C^2)}}{A\sqrt{2(M-2BC)}}\gamma_1,\ q=-\frac{\sqrt{\mu[(2AB-M)(M-2BC)]}}{B\sqrt{2(M-2AC)}}\gamma_2,$$
$$r=\frac{\sqrt{\mu(A^2B^2-(A-B)^2C^2)}}{C\sqrt{2(2AB-M)}}\gamma_3; \tag{9}$$

$$iii)\quad p=\frac{\sqrt{\mu(\sqrt{M^2-3LN}+3AB-M)(\sqrt{M^2-3LN}+3AC-M)}}{A\sqrt{3(\sqrt{M^2-3LN}+3BC-M)}}\gamma_1,$$
$$q=\frac{\sqrt{\mu(\sqrt{M^2-3LN}+3AB-M)(\sqrt{M^2-3LN}+3BC-M)}}{B\sqrt{3(\sqrt{M^2-3LN}+3AC-M)}}\gamma_2,$$
$$r=\frac{\sqrt{\mu(\sqrt{M^2-3LN}+3AC-M)(\sqrt{M^2-3LN}+3BC-M)}}{C\sqrt{3(\sqrt{M^2-3LN}+3AB-M)}}\gamma_3. \tag{10}$$

Equations (8)-(10) represent peculiar invariant manifolds. The integral K and the corresponding enveloping integral assume a stationary value on these manifolds.

It is possible to apply Lyapunov's second method to investigate stability of peculiar invariant manifolds (8)-(10). The first integral K (2) is considered in the capacity of the Lyapunov function.

For example, in the case of invariant manifolds (9), the 2nd variation of integral K has the form

$$\delta^2 K = -\frac{A(A(B+C)-BC)}{4BC}\eta_1^2 - \frac{B(A(B-C)+BC)}{4AC}\xi_1^2 - \frac{C(C(A+B)-AB)}{4AB}\xi_3^2.$$

Here η_1, ξ_1, and ξ_2 are deviations from undisturbed values of the variables p, q, r. The quadratic form is sign-definite with respect to all variables for

$$B > C \text{ and } \frac{BC}{B+C} < A < \frac{BC}{B-C}.$$

Due to Zubov's theorem [2], the invariant manifold (9) is stable.

Euler's Equations in Lie Algebras

Consider more complex (in the aspect of computations) problem: the dynamic system [3] in Lie algebra. Differential equations of this system write:

$$\begin{aligned}
\dot{s}_1 &= -\alpha^2 r_2 r_3 + \alpha r_1 s_2 - (\beta r_3 - s_2)(\beta r_2 + s_3),\\
\dot{s}_2 &= (\alpha^2 + \beta^2) r_1 r_3 - (\alpha r_1 + \beta r_2) s_1 + (\alpha r_3 - s_1) s_3,\\
\dot{s}_3 &= (\beta r_1 - \alpha r_2) s_3,\\
\dot{r}_1 &= r_2(\alpha r_1 + \beta r_2 + 2 s_3) - r_3 s_2 - ((\alpha^2 + \beta^2) r_3 s_2 + \beta s_3^2) x,\\
\dot{r}_2 &= r_3 s_1 - r_1(\alpha r_1 + \beta r_2 + 2 s_3) + ((\alpha^2 + \beta^2) r_3 s_1 + \alpha s_3^2) x,\\
\dot{r}_3 &= r_1 s_2 - r_2 s_1 + (\beta s_1 - \alpha s_2) s_3 x.
\end{aligned} \qquad (11)$$

These equations have the following first integrals:

$$\begin{aligned}
&2V_0 = (s_1^2 + s_2^2 + 2 s_3^2) + 2(\alpha r_1 + \beta r_2) s_3 - (\alpha^2 + \beta^2) r_3^2 = 2h,\\
&V_1 = s_1 r_1 + s_2 r_2 + s_3 r_3 = c_1, \ V_2 = x(s_1^2 + s_2^2 + s_3^2) + r_1^2 + r_2^2 + r_3^2 = c_2,\\
&V_3 = (r_1 s_1 + r_2 s_2)((\alpha^2 + \beta^2)(r_1 s_1 + r_2 s_2) + 2(\alpha s_1 + \beta s_2) s_3) + s_3^2(s_1^2 + s_2^2\\
&\quad + (\alpha r_1 + \beta r_2 + s_3)^2) + x s_3^2 (\beta s_1 - \alpha s_2)^2 = c_3.
\end{aligned}$$

Here s_i and r_i are components of two three-dimensional vectors, α, β, and x are arbitrary constants.
The cases, when $x > 0$ and $x < 0$, correspond to Euler's equations in Lie algebras $so(4)$ and $so(3,1)$, respectively. When $x = 1$, equations (11) coincide with the Poincare–Zhukovsky equations, which describe the motion of a rigid body having an ellipsoidal cavity filled with vortex incompressible fluid. When $x = 0$, system (11) corresponds to the integrable case in Kirchhoff's problem [4].
Let us find the families of invariant manifolds of equations (11) and investigate their peculiar properties.

Finding Invariant Manifolds

We shall use Routh–Lyapunov's method to find the families of invariant manifolds of equations (11). According to this method, some combinations are formed from the problem's first integrals. These combinations may be both linear and nonlinear. Here we have used linear combinations of first integrals:

$$K = \lambda_0 V_0 - \lambda_1 V_1 - \frac{\lambda_2}{2} V_2 - \frac{\lambda_3}{2} V_3 \;\; (\lambda_i = const). \tag{12}$$

Here $\lambda_0, \lambda_1, \lambda_2$, and λ_3 are some constants, which may assume also zero values. The conditions of stationarity for K with respect to variables s_1, s_2, s_3, r_1, r_2, and r_3 write:

$$\begin{aligned}
\partial K/\partial s_1 &= \lambda_0 s_1 - \lambda_1 r_1 - \lambda_3[(\alpha^2+\beta^2) r_1 (r_1 s_1 + r_2 s_2) + s_1 s_3(\alpha r_2 + \beta r_1) \\
&\quad + s_2 s_3(2\alpha r_1 + s_3)] - x(\lambda_3 \beta s_3^2(\beta s_1 - \alpha s_2) + \lambda_2 s_1) = 0, \\
\partial K/\partial s_2 &= \lambda_0 s_2 - \lambda_1 r_2 - \lambda_3[(\alpha^2+\beta^2) r_2 (r_1 s_1 + r_2 s_2) + s_1 s_3(\alpha r_2 + \beta r_1) \\
&\quad + s_2 s_3(2\beta r_2 + s_3)] - x(\lambda_3 \alpha s_3^2(\alpha s_2 - \beta s_1) + \lambda_2 s_2) = 0, \\
\partial K/\partial s_3 &= \lambda_0(\alpha r_1 + \beta r_2 + 2 s_3) - \lambda_1 r_3 - \lambda_3[(r_1 s_1^2 \alpha s_1 + \beta s_2)(s_1 r_1 + s_2 r_2) \\
&\quad + s_3(\alpha r_1 + \beta r_2)^2 + s_3(s_1^2 + s_2^2 + 2 s_3^2) + 3 s_3^2(\alpha r_1 + \beta r_2)] \\
&\quad - x s_3(\lambda_3(\beta s_1 - \alpha s_2)^2 + \lambda_2) = 0, \\
\partial K/\partial r_1 &= \lambda_0 \alpha s_3 - \lambda_1 s_1 - \lambda_2 r_1 - \lambda_3[(\alpha^2+\beta^2) s_1(r_1 s_1 + r_2 s_2) + \alpha s_3^2 \\
&\quad \times(\beta r_2 + \alpha r_1) + \alpha s_3(s_1^2 + s_2^2) + \beta \lambda_3 s_1 s_2 s_3] = 0, \\
\partial K/\partial r_2 &= -\lambda_0 \beta s_3 + \lambda_1 s_2 + \lambda_2 r_2 + \lambda_3[(\alpha^2+\beta^2) s_2(s_1 r_1 + r_2 s_2) + s_2 s_3 \\
&\quad \times(\alpha s_1 + \beta s_2) + \beta s_3^2(\alpha r_1 + \beta r_2) + \beta s_3^3] = 0, \\
\partial K/\partial r_3 &= -((\alpha^2+\beta^2)\lambda_0 + \lambda_2) r_3 + \lambda_1 s_3 = 0.
\end{aligned} \tag{13}$$

The conditions of existence of invariant manifolds for equations (11) may be obtained by equating the Jacobian of the latter system to zero. The solutions of the system found under these conditions will be the desired invariant manifolds. In this problem the Jacobian expression is rather bulky, so its complete analysis is problematic even with the use of computer algebra tools. Some conditions can be easily obtained from equations (13).

When $\lambda_1 = 0, \lambda_2 = -(\alpha^2+\beta^2)\lambda_0$ the last equation of system (13) turns zero. The system becomes degenerate, and, consequently, the system's Jacobian is zero. Having substituted these values for λ_1 and λ_2 into (13) and using the method of Gröbner bases, we have found a series of families of invariant manifolds. You can see below some of these families:

$$
\begin{aligned}
r_1 &= \frac{1}{a_1 a_2(\alpha^2+\beta^2)\lambda_3}\Big(a_3\alpha\lambda_3 s_3 + \sqrt{a_1(x(\alpha^2+\beta^2)+1)\lambda_3}\;((\alpha^2+\beta^2) \\
&\quad \times\sqrt{a_1(\alpha^2+\beta^2)(\lambda_0-\lambda_3 s_3^2)\lambda_0\lambda_3}\; s_1 \mp a_1\alpha\beta\lambda_3 s_3^2)\Big),\\
r_2 &= -\frac{1}{(\alpha^2+\beta^2)\lambda_3}\Big(\beta\lambda_3 s_3 \mp \sqrt{a_1(x(\alpha^2+\beta^2)+1)\lambda_3}\Big),\\
s_2 &= \frac{1}{a_2\lambda_3}\Big(\alpha\beta\lambda_3^2 s_1 s_3^2 \pm \sqrt{a_1(\alpha^2+\beta^2)(\lambda_0-\lambda_3 s_3^2)\lambda_0\lambda_3}\Big).
\end{aligned} \tag{14}
$$

Here $a_1 = (\alpha^2+\beta^2)(\lambda_0-\lambda_3 s_1^2) - \alpha^2\lambda_3 s_3^2$, $a_2 = \alpha^2\lambda_3 s_3^2 - (\alpha^2+\beta^2)\lambda_0$, $a_3 = (\alpha^2+\beta^2)^2(\lambda_0-\lambda_3 s_1^2)\lambda_0 + \alpha^2(\alpha^2+\beta^2)(\lambda_3 s_1^2 - 2\lambda_0)\lambda_3 s_3^2 + \alpha^4\lambda_3^2 s_3^4$.

We have also analyzed the set of solutions of system (13) with the use of the method of Gröbner bases. It has also enabled us to obtain additional families of the invariant manifolds.

The Gröbner basis constructed for the equations (13) writes:

$$
\begin{aligned}
&(a_{139}s_1 + a_{138}s_2 + a_{141}s_3)(a_{135} + a_{136}s_1^2 + a_{136}s_2^2 + a_{106}s_1s_3 + a_{105}s_2s_3 + a_{132}s_3^2) = 0,\\
&s_3(a_{140}s_1 + a_{142}s_3)(a_{135} + a_{136}s_1^2 + a_{136}s_2^2 + a_{106}s_1s_3 + a_{105}s_2s_3 + a_{132}s_3^2) = 0,\\
&\quad s_3(a_{133}s_1^2 + a_{133}s_2^2 + a_{108}s_1s_3 + a_{107}s_2s_3 + a_{110}s_3^2 + a_{128}s_1^2s_3^2 + a_{72}s_1s_2s_3^2\\
&\quad + a_{130}s_2^2s_3^2 + a_{79}s_1s_3^3 + a_{78}s_2s_3^3 + a_{127}s_3^4) = 0,\\
&a_{118}s_1 + a_{117}s_2 + a_{119}s_3 + a_{81}s_1^2s_3 + a_{39}s_1s_2s_3 + a_{80}s_2^2s_3 + a_{62}s_1s_3^2 + a_{124}s_1^3s_3^2\\
&\quad + a_{71}s_2s_3^2 + a_{101}s_1^2s_2s_3^2 + a_{121}s_3^3 + a_{125}s_1^2s_3^3 + a_{34}s_1s_2s_3^3 + a_{102}s_1s_3^4 + a_{77}s_2s_3^4 + a_{126}s_3^5 = 0,\\
&\quad a_{48}s_1 + a_5s_1^3 + a_{63}s_1^5 + a_{18}s_2 + a_6s_1s_2^2 + a_{63}s_1^3s_2^2 + a_{25}s_3 + a_{86}s_1^2s_3 + a_8s_1^4s_3\\
&\quad + a_4s_1s_2s_3 + a_7s_1^3s_2s_3 + a_{85}s_2^2s_3 + a_{111}s_1s_3^2 + a_{92}s_1^3s_3^2 + a_{47}s_2s_3^2 + a_{84}s_3^3\\
&\quad + a_{49}s_1^2s_3^3 + a_2s_1s_2s_3^3 + a_{50}s_1s_3^4 + a_3s_2s_3^4 + a_{88}s_3^5 = 0,\\
&s_3(a_{55}s_1 + a_{56}s_2 + a_{57}s_3 + a_{21}s_1^2s_3 + a_{37}s_1s_2s_3 + a_{38}s_2^2s_3 + a_{120}s_1s_3^2 + a_{123}s_1^3s_3^2\\
&\quad + a_{33}s_2s_3^2 + a_{98}s_3^3 + a_{74}s_1^2s_3^3 + a_{76}s_1s_3^4 + a_{104}s_3^5) = 0,\\
&a_{94}s_1^2 + a_{27}s_1s_2 + a_{54}s_2^2 + a_{53}s_1s_3 + a_{36}s_1^3s_3 + a_{42}s_2s_3 + a_{35}s_1^2s_2s_3 + a_{93}s_3^2\\
&\quad + a_{95}s_1^2s_3^2 + a_{131}s_1^4s_3^2 + a_{29}s_1s_2s_3^2 + a_{99}s_2^2s_3^2 + a_{28}s_1s_3^3 + a_{23}s_2s_3^3 + a_{97}s_3^4\\
&\quad + a_{73}s_1^2s_3^4 + a_{75}s_1s_3^5 + a_{103}s_3^6 = 0,\\
&s_3(a_{122} + a_{114}s_1^2 + a_{114}s_2^2 + a_{41}s_1s_3 + a_{40}s_2s_3 + a_{129}s_3^2 + a_{22}s_1^2s_3^2 + a_{15}s_1s_2s_3^2\\
&\quad + a_{17}s_1s_3^3 + a_{16}s_2s_3^3 + a_{51}s_3^4 + a_{67}s_1^2s_3^4 + a_9s_1s_2s_3^4 + a_{10}s_1s_3^5 + a_1s_2s_3^5 + a_{32}s_3^6) = 0,\\
&a_{69}r_2 + a_{43}s_1 + a_{30}s_1^3 + a_{52}s_2 + a_{30}s_1s_2^2 + a_{44}s_3 + a_{109}s_1^2s_3 + a_{24}s_1s_2s_3\\
&\quad + a_{116}s_2^2s_3 + a_{87}s_1s_3^2 + a_{68}s_1^3s_3^2 + a_{113}s_2s_3^2 + a_{89}s_3^3 + a_{96}s_1^2s_3^3 + a_{11}s_1s_2s_3^3\\
&\quad + a_{60}s_1s_3^4 + a_{13}s_2s_3^4 + a_{46}s_3^5 = 0,\\
&a_{70}r_1 + a_{83}s_1 + a_{65}s_1^3 + a_{19}s_2 + a_{65}s_1s_2^2 + a_{20}s_3 + a_{91}s_1^2s_3 + a_{45}s_1s_2s_3\\
&\quad + a_{82}s_2^2s_3 + a_{112}s_1s_3^2 + a_{100}s_1^3s_3^2 + a_{90}s_2s_3^2 + a_{26}s_3^3 + a_{59}s_1^2s_3^3 + a_{12}s_1s_2s_3^3\\
&\quad + a_{58}s_1s_3^4 + a_{14}s_2s_3^4 + a_{61}s_3^5 = 0,
\end{aligned}
\tag{15}
$$

where a_i $(i = 1, \ldots, 142)$ are some polynomials in λ_i, α, β, and x. The basis has been obtained under elimination monomial order. The time of constructing the basis is 5.71 minutes.

System (15) is decomposed into subsystems. We have analyzed these subsystems separately. A lexicographic Gröbner basis has been constructed for each of the subsystems. As a result, we have found a series of solutions of system (13).

These solutions represent the families of invariant manifolds for the initial system of differential equations. Some of these solutions are given below:

$$\{s_1 = -\frac{\alpha\lambda_1 s_3}{(\alpha^2+\beta^2)\lambda_0},\ s_2 = -\frac{\beta\lambda_1 s_3}{(\alpha^2+\beta^2)\lambda_0},\ r_1 = \frac{\lambda_0 - \lambda_3 s_3(\beta r_2 + s_3)}{\alpha\lambda_3 s_3},$$
$$r_3 = -\frac{\lambda_1 s_3}{(\alpha^2+\beta^2)\lambda_0}\} \ \text{ when } \lambda_2 = 0; \tag{16}$$

$$\{r_1 = \frac{x\alpha s_3\sqrt{x\lambda_1^2 + a\lambda_0^2}}{\sqrt{x\lambda_1^2 + a\lambda_0^2} \mp a\lambda_0}, r_2 = \frac{x\beta s_3\sqrt{x\lambda_1^2 + a\lambda_0^2}}{\sqrt{x\lambda_1^2 + a\lambda_0^2} \mp a\lambda_0}, r_3 = -\frac{x\lambda_1 s_3}{a\lambda_0 \mp \sqrt{x\lambda_1^2 + a\lambda_0^2}},$$
$$s_1 = -\frac{x\alpha\lambda_1 s_3}{a\lambda_0 \mp \sqrt{x\lambda_1^2 + a\lambda_0^2}},\ s_2 = -\frac{x\beta\lambda_1 s_3}{a\lambda_0 \mp \sqrt{x\lambda_1^2 + a\lambda_0^2}}\},$$
$$\text{when } \lambda_2 = \frac{\lambda_0 \mp \sqrt{\mathrm{x}\lambda_1^2 + \mathrm{a}\lambda_0^2}}{\mathrm{x}},\ \lambda_3 = 0. \tag{17}$$

Here $a = (\alpha^2+\beta^2)x + 1$.

Analysis of Peculiar Properties of Invariant Manifolds

Now, let us analyze peculiar properties of the invariant manifolds obtained. We will use the technique of enveloping integral. Consider one of the families of invariant manifolds (17), for example:

$$\{r_1 = \frac{x\alpha s_3\sqrt{x\lambda_1^2 + a\lambda_0^2}}{\sqrt{x\lambda_1^2 + a\lambda_0^2} + a\lambda_0},\ r_2 = \frac{x\beta s_3\sqrt{x\lambda_1^2 + a\lambda_0^2}}{\sqrt{x\lambda_1^2 + a\lambda_0^2} + a\lambda_0},\ r_3 = -\frac{x\lambda_1 s_3}{a\lambda_0 + \sqrt{x\lambda_1^2 + a\lambda_0^2}},$$
$$s_1 = -\frac{x\alpha\lambda_1 s_3}{a\lambda_0 + \sqrt{x\lambda_1^2 + a\lambda_0^2}},\ s_2 = -\frac{x\beta\lambda_1 s_3}{a\lambda_0 + \sqrt{x\lambda_1^2 + a\lambda_0^2}}\},\ \lambda_2 = \frac{\lambda_0 + \sqrt{x\lambda_1^2 + a\lambda_0^2}}{x},$$
$$\lambda_3 = 0. \tag{18}$$

Substitute the values of λ_2, λ_3, which correspond to this family of invariant manifolds, into the integral K (12):

$$K = \lambda_0 V_0 - \lambda_1 V_1 - \frac{\lambda_0 + \sqrt{x\lambda_1^2 + a\lambda_0^2}}{2x} V_2.$$

Next, compute the derivative of the latter expression with respect to λ_1 and equate it to zero:

$$\partial K/\partial\lambda_1 = -V_1 - \frac{\lambda_1}{2\sqrt{x\lambda_1^2 + a\lambda_0^2}} V_2 = 0.$$

After some transformations and after substitution of the respective expressions for the integrals, we have:

$$2\sqrt{x\lambda_1^2 + a\lambda_0^2}(s_1 r_1 + s_2 r_2 + s_3 r_3) + \lambda_1(x(s_1^2 + s_1^2 + s_3^2) + r_1^2 + r_2^2 + r_3^2) = 0. \tag{19}$$

Find the value of this expression on the family of invariant manifolds (18). Substitute the values of the variables from (18) into the latter equality. As a result, the equality shall turn into an identity. This means that the family of invariant manifolds (18) is peculiar. The latter is valid for all the values of the parameter λ_1, for which there exists the invariant manifolds under scrutiny.

Investigation of stability for the family of invariant manifolds (18) has shown that elements of this family are stable in the sense of Lyapunov when $\lambda_0 > 0 \wedge \lambda_1 \neq 0 \wedge x > 0$.

Now consider a case when the family of first integrals, which corresponds to the family of invariant manifolds under investigation, is linear with respect to parameters λ_i. In this case, the derivative of the integral K computed with respect to any of the parameters does not contain the parameter λ_i needed for constructing the enveloping integral. If the family of invariant manifolds under consideration is multiparametric, it is possible to analyze the peculiar properties of one-parameter subfamilies of such invariant manifolds. Consider this problem for the family of invariant manifolds (16):

$$\{s_1 = -\frac{\alpha\lambda_1 s_3}{(\alpha^2+\beta^2)\lambda_0},\ s_2 = -\frac{\beta\lambda_1 s_3}{(\alpha^2+\beta^2)\lambda_0},\ r_1 = \frac{\lambda_0 - \lambda_3 s_3(\beta r_2 + s_3)}{\alpha\lambda_3 s_3},\ r_3 = -\frac{\lambda_1 s_3}{(\alpha^2+\beta^2)\lambda_0}\}$$

when $\lambda_2 = 0$. (20)

Substitute $\lambda_2 = 0$ into K (12). As a result, we have: $K = \lambda_0 V_0 - \lambda_1 V_1 - \lambda_3 V_3/2$. Find a one-parameter subfamily of the family of invariant manifolds (20). We shall consider the parameters λ_0 and λ_3 as functions of λ_1.
Compute the derivative of the latter expression with respect to λ_1 and equate it to zero:

$$\frac{\partial K}{d\lambda_1} = \frac{d\lambda_0}{d\lambda_1}V_0 - V_1 - \frac{1}{2}\frac{d\lambda_3}{d\lambda_1}V_3 = 0. \quad (21)$$

After some transformations, equality (21) writes:

$$\frac{2\lambda_3(\lambda_1)(\lambda_1 + (\alpha^2+\beta^2)\lambda_0(\lambda_1)\dot{\lambda}_0(\lambda_1)) - (\lambda_1^2 + (\alpha^2+\beta^2)\lambda_0^2(\lambda_1))\dot{\lambda}_3(\lambda_1)}{2(\alpha^2+\beta^2)\lambda_3^2(\lambda_1)} = 0,$$

where $\dot{\lambda}_i = d\lambda_i/d\lambda_1$.
Integration of the latter expression gives the following relationship between the parameters:

$$\lambda_3(\lambda_1) = \bar{C}(\lambda_1^2 + (\alpha^2+\beta^2)\lambda_0^2(\lambda_1)),\ \text{where } \bar{C} \text{ is a constant of integration.} \quad (22)$$

The equations of family under consideration, after excluding parameter λ_3 from them with the use of relationship (22), define a subfamily of peculiar invariant manifolds.

The Problem Related to Motion of a System of Vortices

Finally, let us consider the problem of motion of N parallel direct vortex lines (having intensities α_i) in an unbounded volume of an ideal fluid. Motion of such a system is described by Kirchhoff's equations [9]. We will consider the case when $N = 3$. In this case, the equations of motion write:

$$\dot{x}_1 = -\frac{\alpha_2(y_1 - y_2)}{4\pi((x_1 - x_2)^2 + (y_1 - y_2)^2)} - \frac{\alpha_3(y_1 - y_3)}{4\pi((x_1 - x_3)^2 + (y_1 - y_3)^2)},$$

$$\dot{x}_2 = \frac{\alpha_1(y_1 - y_2)}{4\pi((x_1 - x_2)^2 + (y_1 - y_2)^2)} - \frac{\alpha_3(y_2 - y_3)}{4\pi((x_2 - x_3)^2 + (y_2 - y_3)^2)},$$

$$\dot{x}_3 = \frac{\alpha_1(y_1 - y_3)}{4\pi((x_1 - x_3)^2 + (y_1 - y_3)^2)} + \frac{\alpha_2(y_2 - y_3)}{4\pi((x_2 - x_3)^2 + (y_2 - y_3)^2)},$$

$$\dot{y}_1 = \frac{\alpha_2(x_1 - x_2)}{4\pi((x_1 - x_2)^2 + (y_1 - y_2)^2)} + \frac{\alpha_3(x_1 - x_3)}{4\pi((x_1 - x_3)^2 + (y_1 - y_3)^2)},$$

$$\dot{y}_2 = -\frac{\alpha_1(x_1 - x_2)}{4\pi((x_1 - x_2)^2 + (y_1 - y_2)^2)} + \frac{\alpha_3(x_2 - x_3)}{4\pi((x_2 - x_3)^2 + (y_2 - y_3)^2)},$$

$$\dot{y}_3 = -\frac{\alpha_1(x_1 - x_3)}{4\pi((x_1 - x_3)^2 + (y_1 - y_3)^2)} - \frac{\alpha_2(x_2 - x_3)}{4\pi((x_2 - x_3)^2 + (y_2 - y_3)^2)} \qquad (23)$$

Here (x_i, y_i) are the Cartesian coordinates for the points of intersection of vortex lines with the plane perpendicular to them.

The differential equations possess the following first integrals:

$$\begin{aligned} V_0 &= -\frac{1}{8\pi}\Big(\alpha_1\alpha_2 \ln\left((x_1 - x_2)^2 + (y_1 - y_2)^2\right) + \alpha_1\alpha_3 \ln\left((x_1 - x_3)^2 + (y_1 - y_3)^2\right) \\ &+ \alpha_2\alpha_3 \ln\big((x_2 - x_3)^2 + (y_2 - y_3)^2\big)\Big) = h = const, \\ V_1 &= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = c_1 = \text{const}, \ V_2 = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = c_2 = \text{const}, \\ V_3 &= \alpha_1(x_1^2 + y_1^2) + \alpha_2(x_2^2 + y_2^2) + \alpha_3(x_3^2 + y_3^2) = c_3 = \text{const}, \qquad (24) \end{aligned}$$

We shall consider the problem of finding of peculiar invariant manifolds for the equations (23). In this problem one of the first integrals is logarithmic.

Finding the Invariant Manifolds

To find the invariant manifolds for the system (23) we introduce the function

$$K = 4\lambda_0 V_0 - \frac{1}{2}\lambda_1 V_1^2 - \frac{1}{2}\lambda_2 V_2^2 - \frac{1}{2}\lambda_3 V_3, \qquad (25)$$

where $\lambda_0, \lambda_1, \lambda_2$, and λ_3 are some constants.
Write the conditions of stationarity of K with respect to variables x_1, x_2, x_3, y_1, y_2, and y_3:

$$\begin{aligned}
\partial K/\partial x_1 &= -\alpha_1\lambda_0\Big(\frac{\alpha_2(x_1-x_2)}{\pi((x_1-x_2)^2+(y_1-y_2)^2)} + \frac{\alpha_3(x_1-x_3)}{\pi((x_1-x_3)^2+(y_1-y_3)^2)}\Big) \\
&\quad -\alpha_1\lambda_1(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3) - \alpha_1\lambda_3x_1 = 0, \\
\partial K/\partial x_2 &= \alpha_2\lambda_0\Big(\frac{\alpha_1(x_1-x_2)}{\pi((x_1-x_2)^2+(y_1-y_2)^2)} - \frac{\alpha_3(x_2-x_3)}{\pi((x_2-x_3)^2+(y_2-y_3)^2)}\Big) \\
&\quad -\alpha_2\lambda_1(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3) - \alpha_2\lambda_3x_2 = 0, \\
\partial K/\partial x_3 &= \alpha_3\lambda_0\Big(\frac{\alpha_1(x_1-x_3)}{\pi((x_1-x_3)^2+(y_1-y_3)^2)} + \frac{\alpha_2(x_2-x_3)}{\pi((x_2-x_3)^2+(y_2-y_3)^2)}\Big) \\
&\quad -\alpha_3\lambda_1(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3) - \alpha_3\lambda_3x_3 = 0, \\
\partial K/\partial y_1 &= -\alpha_1\lambda_0\Big(\frac{\alpha_3(y_1-y_3)}{\pi((x_1-x_3)^2+(y_1-y_3)^2)} + \frac{\alpha_2(y_1-y_2)}{\pi((x_1-x_2)^2+(y_1-y_2)^2)}\Big) \\
&\quad -\alpha_1\lambda_2(\alpha_1y_1+\alpha_2y_2+\alpha_3y_3) - \alpha_1\lambda_3y_1 = 0, \\
\partial K/\partial y_2 &= \alpha_2\lambda_0\Big(\frac{\alpha_1(y_1-y_2)}{\pi((x_1-x_2)^2+(y_1-y_2)^2)} - \frac{\alpha_3(y_2-y_3)}{\pi((x_2-x_3)^2+(y_2-y_3)^2)}\Big) \\
&\quad -\alpha_2\lambda_2(\alpha_1y_1+\alpha_2y_2+\alpha_3y_3) - \alpha_2\lambda_3y_2 = 0, \\
\partial K/\partial y_3 &= \alpha_3\lambda_0\Big(\frac{\alpha_1(y_1-y_3)}{\pi((x_1-x_3)^2+(y_1-y_3)^2)} + \frac{\alpha_2(y_2-y_3)}{\pi((x_2-x_3)^2+(y_2-y_3)^2)}\Big) \\
&\quad -\alpha_3\lambda_2(\alpha_1y_1+\alpha_2y_2+\alpha_3y_3) - \alpha_3\lambda_3y_3 = 0. \qquad (26)
\end{aligned}$$

The desired invariant manifolds can be obtained as solutions of the latter system.
In order to obtain the conditions for existence of such solutions of system (26) we reduce the system to the form:

$$\alpha_1\lambda_0\Big(\frac{\alpha_2(x_1-x_2)}{\pi((x_1-x_2)^2+(y_1-y_2)^2)}+\frac{\alpha_3(x_1-x_3)}{\pi((x_1-x_3)^2+(y_1-y_3)^2)}\Big)+\alpha_1\lambda_1(\alpha_1x_1+\alpha_2x_2$$
$$+\alpha_3x_3)+\alpha_1\lambda_3x_1=0,$$
$$\alpha_2\lambda_0\Big(\frac{\alpha_1(x_1-x_2)}{\pi((x_1-x_2)^2+(y_1-y_2)^2)}-\frac{\alpha_3(x_2-x_3)}{\pi((x_2-x_3)^2+(y_2-y_3)^2)}\Big)-\alpha_2\lambda_1(\alpha_1x_1+\alpha_2x_2$$
$$+\alpha_3x_3)-\alpha_2\lambda_3x_2=0,$$
$$(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3)((\alpha_1+\alpha_2+\alpha_3)\lambda_1+\lambda_3)=0,$$
$$\alpha_1\lambda_0\Big(\frac{\alpha_3(y_1-y_3)}{\pi((x_1-x_3)^2+(y_1-y_3)^2)}+\frac{\alpha_2(y_1-y_2)}{\pi((x_1-x_2)^2+(y_1-y_2)^2)}\Big)+\alpha_1\lambda_2(\alpha_1y_1+\alpha_2y_2$$
$$+\alpha_3y_3)+\alpha_1\lambda_3y_1=0,$$
$$\alpha_2\lambda_0\Big(\frac{\alpha_1(y_1-y_2)}{\pi((x_1-x_2)^2+(y_1-y_2)^2)}-\frac{\alpha_3(y_2-y_3)}{\pi((x_2-x_3)^2+(y_2-y_3)^2)}\Big)-\alpha_2\lambda_2(\alpha_1y_1+\alpha_2y_2$$
$$+\alpha_3y_3)-\alpha_2\lambda_3y_2=0,$$
$$(\alpha_1y_1+\alpha_2y_2+\alpha_3y_3)((\alpha_1+\alpha_2+\alpha_3)\lambda_2+\lambda_3)=0. \qquad (27)$$

The third and the last of the equations turn zero, when one of the following conditions hold:

$$\text{i) } \lambda_1=-\frac{\lambda_3}{\alpha_1+\alpha_2+\alpha_3},\ \lambda_2=-\frac{\lambda_3}{\alpha_1+\alpha_2+\alpha_3};\ \text{ii) } \lambda_1=-\frac{\lambda_3}{\alpha_1+\alpha_2+\alpha_3};$$
$$\text{iii) } \lambda_2=-\frac{\lambda_3}{\alpha_1+\alpha_2+\alpha_3}. \qquad (28)$$

In these cases, the system will be degenerate, and, consequently, it will have invariant manifolds in the capacity of solutions. Conditions (28) are considered as necessary conditions of existence of invariant manifolds for differential equations (23).

Substitute one of the conditions (28) into equations (27). These equations will be reduced to a cubic equation. Solutions of the equation can be easily obtained with the use of *Mathematica* tools, but these will be rather bulky. We have obtained solutions of the cubic equation under various restrictions imposed on the parameters α_i.

You can see below some of the families of solutions obtained:

$$\frac{1}{2}a_1\alpha_3^2(x_1-x_2)^2 - a_0(y_2-y_3)^2 - \frac{\alpha_1\alpha_3^2\lambda_0(a_2\alpha_1+\alpha_3^2)}{\pi\lambda_3} = 0,$$
$$(x_3-x_2)^2-(y_2-y_3)^2+\frac{\alpha_1\alpha_3^2(\alpha_3+\sqrt{4\alpha_1^2-3\alpha_3^2})\lambda_0}{\pi\lambda_3(\alpha_1-\alpha_3)a_1} = 0,$$
$$2\alpha_3y_1+(a_2-\alpha_3)y_2-(a_2+\alpha_3)y_3=0 \text{ when } \lambda_1=\lambda_2=-\frac{\lambda_3}{\alpha_1},\ \alpha_2=-\alpha_3; \quad (29)$$

$$\frac{\alpha_2(x_1-x_2)^2}{2\alpha_2+\alpha_3}+\frac{2\alpha_2+\alpha_3}{5\alpha_2+4\alpha_3}y_3^2+\frac{\alpha_2^2\lambda_0}{\pi\lambda_3(\alpha_2+\alpha_3)}=0,$$
$$\frac{\alpha_2^2a_4(x_3-x_2)^2}{2\alpha_2+\alpha_3}+\frac{a_3(2\alpha_2+\alpha_3)(\alpha_2+\alpha_3)^2}{5\alpha_2+4\alpha_3}y_3^2+\frac{a_3(\alpha_2+\alpha_3)\alpha_2^2\lambda_0}{\lambda_3\pi}=0,$$
$$2\alpha_2\sqrt{5\alpha_2+4\alpha_3}y_1+a_5y_3=0,\ 2\alpha_2\sqrt{5\alpha_2+4\alpha_3}y_2+a_4y_3=0$$
$$\text{when } \lambda_1=-\frac{\lambda_3}{2\alpha_2+\alpha_3},\ \alpha_1=\alpha_2. \quad (30)$$

$$\alpha_3^2x_3^2+\alpha_2^2(y_1-y_2)^2+\frac{\alpha_2^4\alpha_3(\alpha_2-a_6)\lambda_0}{\pi\lambda_3a_7}=0,$$
$$\alpha_3^2a_7x_3^2+\frac{\alpha_2^4((a_6+\alpha_2)\alpha_3-2\alpha_2^2)}{\alpha_2+\alpha_3}(y_3-y_2)^2+\frac{\alpha_2^4\alpha_3(\alpha_2-a_6)\lambda_0}{\lambda_3\pi}=0,$$
$$2\alpha_2^2x_1+((a_6-\alpha_2)\alpha_3-2\alpha_2^2)x_3=0,\quad 2\alpha_2^3x_2+\alpha_2((a_6+\alpha_2)\alpha_3-2\alpha_2^2)x_3=0$$
$$\text{when } \lambda_2=-\frac{\lambda_3}{\alpha_3},\ \alpha_1=-\alpha_2. \quad (31)$$

Here a_i $(i=1,\dots,7)$ are some expressions of α_j $(j=1,2,3)$.
These solutions represent the families of invariant manifolds for the initial differential equations.

Analysis of Peculiar Properties of Invariant Manifolds

Here we have also employed our technique for the analysis of peculiar properties of the families of invariant manifolds. Consider, for example, the family of invariant manifolds (29):

$$\frac{1}{2}a_1\alpha_3^2(x_1-x_2)^2 - a_0(y_2-y_3)^2 - \frac{\alpha_1\alpha_3^2\lambda_0(a_2\alpha_1+\alpha_3^2)}{\pi\lambda_3} = 0,$$

$$(x_3-x_2)^2-(y_2-y_3)^2+\frac{\alpha_1\alpha_3^2(\alpha_3+\sqrt{4\alpha_1^2-3\alpha_3^2})\lambda_0}{\pi\lambda_3(\alpha_1-\alpha_3)a_1} = 0,$$

$$2\alpha_3y_1+(a_2-\alpha_3)y_2-(a_2+\alpha_3)y_3 = 0 \text{ when } \lambda_1=\lambda_2=-\frac{\lambda_3}{\alpha_1},\ \alpha_2=-\alpha_3. \quad (32)$$

Substitute the values of parameters λ_1 and λ_2 into K (25). As a result, we have

$$2\tilde{K} = -8\lambda_0V_0+\frac{\lambda_3}{\alpha_1}V_1^2+\frac{\lambda_3}{\alpha_1}V_2^2-\lambda_3V_3.$$

The latter expression represents a combination of first integrals, which is linear with respect to the parameters. The partial derivative $\tilde{K}$ computed with respect to λ_3 (or λ_0) does not contain the parameter needed for constructing the enveloping integral.

Find the relationship between the parameters λ_0, λ_3, under which the subfamily of invariant manifolds (32) is peculiar.

We will consider the parameter λ_3 as a function of λ_0. Compute the partial derivative of $\tilde{K}$ with respect to λ_0:

$$2\frac{\partial\tilde{K}}{\partial\lambda_3} = -8V_0+\frac{d\lambda_3}{\alpha_1 d\lambda_0}V_1^2+\frac{d\lambda_3}{\alpha_1 d\lambda_0}V_2^2-\frac{d\lambda_3}{d\lambda_0}V_3 = 0.$$

After some transformations, the latter expression writes:

$$\frac{\alpha_3^2\lambda_0\dot{\lambda}_3(\lambda_0)}{2\pi\lambda_3(\lambda_0)} - \ln\Big(2^{-\alpha_1\alpha_3/2\pi}\pi^{-\alpha_3^2/2\pi}\Big(\frac{2\alpha_1^2-\alpha_3(\alpha_3+\sqrt{4\alpha_1^2-3\alpha_3^2})}{(\alpha_1+\alpha_3)^2}\Big)^{\alpha_1\alpha_3/2\pi}$$

$$\times\Big(-\frac{\alpha_1\alpha_3^2\lambda_0}{(\alpha_3^2-\alpha_1(2\alpha_1-\sqrt{4\alpha_1^2-3\alpha_3^2}))\lambda_3(\lambda_0)}\Big)^{\alpha_3^2/2\pi}\Big) = 0.$$

The expression obtained will be considered as a differential equation with respect to λ_3. Using the *Mathematica* function "DSolve", we have found the solution of this equation:

$$\lambda_3(\lambda_0) = -\frac{\left(e^{\frac{1}{\alpha_3\lambda_0}e^{C[1]\alpha_3}-1}\alpha_1\alpha_3^2(\alpha_1+\alpha_3)^{-2\alpha_1/\alpha_3}\left(\alpha_1^2-\frac{1}{2}\alpha_3(\alpha_3+\sqrt{4\alpha_1^2-3\alpha_3^2}\ )\right)^{\alpha_1/\alpha_3}\lambda_0\right)}{\pi(\alpha_3^2-\alpha_1(2\alpha_1-\sqrt{4\alpha_1^2-3\alpha_3^2})},$$

where $C[1]$ is a constant of integration.
So, when the latter relationship between the parameters λ_0 and λ_3 takes place, the corresponding subfamily of the family of invariant manifolds (32) is peculiar. Similar results have been obtained for the families of invariant manifolds (30), (31).

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