Deducing the constraints in the light-cone $SU(3)$ Yang-Mills mechanics via Gröbner bases

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Contents

Contents

Degenerate Lagrangian Systems

Modern theories of gravity and elementary particle physics contain gauge degrees of freedom and by this reason are described by degenerate Lagrangians.

In mechanics: Lagrangian $L(q,\dot{q})$ is a function of (generalized) coordinates $q := q_1, q_2, \ldots, q_n$ and velocities $\dot{q} := \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$.

The Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0\,, \qquad 1 \leq i \leq n$$

have the structure

$$H_{ij}\ddot{q}_j + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\,\dot{q}_j - \frac{\partial L}{\partial q_i} = 0\,, \qquad H_{ij} := \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

Degenerate Lagrangian Systems

Lagrangian $L(q,\dot q)$ is

1. regular if $r := rank\|H_{ij}\| = n$
2. degenerate (singular) if $r < n$

In the 1st case the Euler-Lagrange equations are solved with respect to the accelerations ($\ddot q$), and there is no hidden constraints.

In the 2nd case the equations cannot be solved with respect to all accelerations, and there are $n-r$ functionally independent constraints

$$\varphi_\alpha(q,\dot q) = 0, \qquad 1 \le \alpha \le n-r$$

If these constraints cannot be integrated (reduced to ones depending on the coordinates only), the mechanics is nonholonomic.

Remark. If Lagrangian $L_0(q,\dot q)$ is regular with externally imposed holonomic constraints $\varphi_\alpha(q) = 0$, the system is equivalent to the singular one with Lagrangian $L = L_0 + \lambda_\alpha\varphi_\alpha$ and extra generalized coordinates λ_α.

Contents

Dirac's Hamiltonian Formalism

Aimed at quantisation of gauge systems.

Passing to the Hamiltonian description via a Legendre transformation

$$p_i := \frac{\partial L}{\partial \dot{q}_i}$$

the degeneracy of the Hessian H_{ij} manifests itself in the existence of $n - r$ relations between coordinates and momenta, the set Σ_1 of primary constraints

$$\Sigma_1 := \{ \phi^{(1)}_\alpha(p, q) = 0 \mid 1 \leq \alpha \leq n - r \} .$$

The dynamics is constrained by the set Σ_1 and is governed by the total Hamiltonian

$$H_T := H_C + U_\alpha \phi^{(1)}_\alpha ,$$

where $H_C(p, q) := p_i q_i - L$ is the canonical Hamiltonian and U_α are Lagrange multipliers.

Consistency Conditions

Hamiltonian equations are given by

$$\dot{q}_i = \{H_T, q_i\}, \quad \dot{p}_i = \{H_T, p_i\}, \quad \phi_\alpha^{(1)}(p, q) = 0$$

with Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial p}{\partial q_i}$$

The primary constraints must satisfy the consistency conditions

$$\dot{\phi}_\alpha^{(1)} = \{H_T, \phi_\alpha^{(1)}\} \stackrel{\Sigma_1}{=} 0 \quad (1 \leq \alpha \leq n - r)$$

$\stackrel{\Sigma_1}{=}$ means the equality modulo the set of primary constraints.

Complete Set of Constraints

The consistency condition for $\phi_\alpha^{(1)}(p,q)$, unless it is satisfied identically, lead to one of the alternatives:

1. Contradiction $\Longleftrightarrow$ inconsistency.
2. New constraint. If it does not involve U_α, it is called secondary constraint and must be added to the constraint set.

The iteration of the consistency check ends up with the complete set of constraints

$$\Sigma := \{\ \phi_\alpha(p,q) = 0\ \mid\ 1 \leq \alpha \leq k\ \}$$

which contains primary $\phi_\alpha^{(1)}(p,q)$, secondary $\phi_\alpha^{(2)}(p,q)$, ternary $\phi_\alpha^{(3)}(p,q)$, quaternary $\phi_\alpha^{(4)}(p,q)$, etc., constraints.

Remark. Secondary, etc., constraints are integrability conditions of the Hamiltonian system, and their incorporation is completion to involution (Hartley, Tucker, Seiler)

Constraints of First and Second Classes

The co-rank $s := k - rank(\mathbb{M})$ of the Poisson bracket matrix

$$\mathbb{M}_{\alpha\beta} :\stackrel{\Sigma}{=} \{\phi_\alpha, \phi_\beta\}\,,$$

represent the number of first-class constraints $\psi_1\,, \psi_2\,, \dots\,, \psi_s$.
Generally, they are linear combinations of constraints ϕ_α

$$\psi_\alpha(p, q) = \sum_\beta \mathrm{c}_{\alpha\beta}(p, q)\,\phi_\beta\,,$$

whose Poisson brackets are zero modulo the constraints set

$$\{\psi_\alpha(p, q), \psi_\beta(p, q)\} \stackrel{\Sigma}{=} 0 \qquad 1 \leq \alpha\,, \beta \leq s\,.$$

The remaining functionally independent constraints form the subset of second-class constraints.

Gauge Transformations

First-class constraints play a very special role in the Hamiltonian description: they generate gauge symmetry.

By Dirac's conjecture, the generator G of gauge transformations is expressed as a linear combination of the first-class constraints

$$G = \sum_{\alpha=1}^{s} \varepsilon_\alpha \psi_\alpha(p, q)$$

where the coefficients ε_α are functions of t.

The generator G must be conserved modulo the primary constraints

$$\frac{dG}{dt} \stackrel{\Sigma_1}{=} 0$$

and its action on phase space coordinates (p, q), in the presence of the first-class constraints only, is given by

$$\delta q_i = \{G, q_i\}, \qquad \delta p_i = \{G, p_i\}\,.$$

Physical Observables

Physical requirement: observables are invariant (singlets) under the gauge symmetry transformations.

This requirement has direct impact on the Hamiltonian reduction, that is a formulation of a new Hamiltonian system with a reduced number of degrees of freedom but equivalent to the initial degenerate one.

The presence of s first-class constraints and $r := k - s$ second-class constraints guarantees the possibility of local reformulation of the initial $2n$ dimensional Hamiltonian system as a $2n - 2s - r$ dimensional reduced (unconstrained) Hamiltonian system.

Remark. The reduced Hamiltonian system admits the canonical quantisation by imposing the standard commutation relations on the phase space variables.

Algorithmisation Issues

1. Compute all primary constraints
2. Determine all integrability conditions (secondary constraints) and separate them into first and second classes.
3. Construct the gauge symmetries generator and the basis for singlet observables
4. Find an equivalent unconstrained Hamiltonian system on the reduced phase space

Assumption. Hereafter we consider dynamical systems whose Lagrangians are polynomials in coordinates and velocities with rational (possibly parametric) coefficients

$$L(q, \dot{q}) \in \mathbb{Q}[q, \dot{q}]$$

Under this assumption issues I-II and the first part of issue III admit the complete algorithmisation.

Contents

Primary Constraints and Canonical Hamiltonian: algorithm

1. Use relations $p_i := \partial L/\partial \dot{q}_i$ as generators of polynomial ideal in $\mathbb{Q}[p, q, \dot{q}]$

$$I_{p,q,\dot{q}} := \mathrm{Id}(\cup_{i=1}^{n}\{p_i - \partial L/\partial \dot{q}_i\}) \subset \mathbb{Q}[p, q, \dot{q}]$$

2. Construct Gröbner basis (Buchberger) or involutive basis (Gerdt,Blinkov) $GB(I_{p,q,\dot{q}})$ by using an appropriate term ordering which eliminates $\dot{q}$, and take the intersection

$$GB(I_{p,q}) = GB(I_{p,q,\dot{q}}) \cap \mathbb{Q}[p, q]$$

3. Extract a subset $\Phi_1 \subset GB(I_{p,q})$ of algebraically independent primary constraints satisfying

$$\forall \phi(p, q) \in \Phi_1 \; : \; \phi(p, q) \notin \mathrm{Id}(\Phi_1 \setminus \{\phi(p, q)\})$$

that is verified by the normal form $NF(\phi, GB(\mathrm{Id}(\Phi_1 \setminus \{\phi\}))$.

4. Compute $H_c(p, q) = NF(p_i q_i - L, GB(I_{p,q,\dot{q}}))$.

Contents

Complete Set of Constraints: algorithm

1. Compute Gröbner (involutive) basis *GB* of the ideal $\mathrm{Id}(\Psi) \subset Q[p,q]$ generated by $\Psi := \Phi_1$ in with respect to some ordering. Fix this ordering in the sequel.
2. Construct the total Hamiltonian $H_T = H_c + U_\alpha \phi_\alpha^{(1)}$ with Lagrange multipliers U_α treated as symbolic constants (parameters).
3. For every element $\phi_\alpha \in \Psi$ compute $h := NF(\{H_T, \phi_\alpha\}, GB)$. If $h \neq 0$ and no multipliers U_β occur in h, then enlarge set Ψ with h, and compute the Gröbner (involutive) basis *GB* for the enlarged set.
4. If $GB = \{1\}$, stop because the system is inconsistent. Otherwise, repeat the previous step until the consistency condition is satisfied for every element in Ψ irrespective of multipliers U_α.
5. Extract algebraically independent set $\Phi = \{\phi_1, \ldots, \phi_k\}$ from *GB*. This gives the complete set of constraints.

Contents

Separation of Constraints: algorithm

1. Construct the $k \times k$ Poisson bracket matrix as

$$\mathbb{M}_{\alpha,\beta} := NF(\{\phi_\alpha, \phi_\beta\}, GB)$$

2. Compute rank r of M.
   If $r = k$, stop with $\Phi_1 = \emptyset$, $\Phi_2 = \Phi$.
   If $r = 0$, stop with $\Phi_1 = \Phi$ and $\Phi_2 = \emptyset$.
   Otherwise, go to the next step.
3. Find a basis $A = \{a_1, \ldots, a_{k-r}\}$ of the null space (kernel) of $\mathbb{M}$. For every $a \in A$ construct a first-class constraint as $a_\alpha \phi_\alpha$. Collect them in set Φ_1.
4. Construct $(k - r) \times k$ matrix $(a_j)_\alpha$ from components of vectors in A and find a basis $B = \{b_1, \ldots, b_r\}$ of the null space of the corresponding linear transformation (cokernel of $\mathbb{M}$) . For every $b \in B$ construct a second-class constraint as $b_\alpha \phi_\alpha$. Collect them in set Φ_2.

Contents

Light-Cone Yang-Mills Mechanics

Lagrangian is given by

$$L := \frac{1}{2g^2}\,\left(F^a_{+-}\,F^a_{+-} + 2\,F^a_{+k}\,F^a_{-k} - F^a_{12}\,F^a_{12}\right)\,.$$

Here g is the “renormalized” coupling constant, and

$$\begin{aligned}
F^a_{+-} &:= \frac{\partial A^a_-}{\partial x^+} + \mathrm{f}^{abc}\,A^b_+\,A^c_-\,,\\
F^a_{+k} &:= \frac{\partial A^a_k}{\partial x^+} + \mathrm{f}^{abc}\,A^b_+\,A^c_k\,,\\
F^a_{-k} &:= \mathrm{f}^{abc}\,A^b_-\,A^c_k\,,\\
F^a_{ij} &:= \mathrm{f}^{abc}\,A^b_i\,A^c_j\,, \quad i,j,k = 1,2
\end{aligned}$$

where $A^a = A^a(x^+)$ $(a = 1, 2, \ldots, n^2-1)$, $x^+ := \frac{1}{\sqrt{2}}\left(x^0 + x^3\right)$, and f^{abc} are the structure constants of $SU(n)$.

Hamiltonian Formulation

The Legendre transformation

$$\pi_a^+ := \frac{\partial L}{\partial \dot{A}_+^a} = 0\,,$$

$$\pi_a^- := \frac{\partial L}{\partial \dot{A}_-^a} = \frac{1}{g^2}\left(\dot{A}_-^a + \mathrm{f}^{abc} A_+^b A_-^c\right)\,,$$

$$\pi_a^k := \frac{\partial L}{\partial \dot{A}_k^a} = \frac{1}{g^2}\,\mathrm{f}^{abc} A_-^b A_k^c$$

gives the canonical Hamiltonian

$$H_C = \frac{g^2}{2}\,\pi_a^-\pi_a^- - \mathrm{f}^{abc} A_+^b \left(A_-^c\,\pi_a^- + A_k^c\,\pi_a^k\right) + \frac{1}{2g^2} F_{12}^a F_{12}^a\,.$$

The non-vanishing Poisson brackets between the canonical variables

$$\{A_\pm^a\,,\pi_b^\pm\} = \delta_b^a\,, \qquad \{A_k^a\,,\pi_b^l\} = \delta_k^l \delta_b^a$$

Primary and Some Secondary Constraints

det $||\frac{\partial^2 L}{\partial\dot{A}\partial\dot{A}}|| = 0$, and the primary constraints are

$$\begin{cases} \varphi_a^{(1)} := \pi_a^+ = 0 \\ \chi_k^a := g^2\,\pi_k^a + \mathrm{f}^{abc} A_-^b A_k^c = 0 \end{cases} \qquad \{\chi_i^a\,, \chi_j^b\} = 2\mathrm{f}^{abc}\eta_{ij}A_-^c$$

The total Hamiltonian $H_T := H_C + U_a\varphi_a^{(1)} + V_k^a\chi_k^a$ yields for $\varphi_a^{(1)}$

$$\dot{\varphi}_a^{(1)} = \{\pi_a^+\,, H_T\} = \mathrm{f}^{abc}\left(A_-^b\pi_c^- + A_k^b\pi_c^k\right) \stackrel{\Sigma_1}{=} 0$$

that generates $n^2 - 1$ secondary constraints

$$\varphi_a^{(2)} := \mathrm{f}_{abc}\left(A_-^b\pi_c^- + A_k^b\pi_c^k\right) = 0, \quad \{\varphi_a^{(2)}\,, \varphi_b^{(2)}\} = \mathrm{f}_{abc}\,\varphi_c^{(2)}$$

The same procedure for χ_k^a gives the consistency conditions

$$\dot{\chi}_k^a = \{\chi_k^a\,, H_C\} - 2\,g^2\,\mathrm{f}^{abc}\,V_k^b\,A_-^c \stackrel{\Sigma_1}{=} 0$$

The further analysis depends on n.

Contents

Constraints and Their Separation

For $SU(2)$: $\mathrm{f}^{abc} := \epsilon^{abc}$. The complete set of constraints contains 9 primary constraints $\varphi_a^{(1)}, \chi_k^a$ and 3 secondary ones $\varphi_a^{(2)}$. Separation of the primary constraints gives 2 additional first-class constraints

$$\psi_k := A_-^a \chi_k^a \,,$$

and 4 second-class constraints

$$\chi_{k\perp}^a := \chi_k^a - \frac{\left(A_-^b \chi_k^b\right)\, A_-^a}{(A_-^1)^2 + (A_-^2)^2 + (A_-^3)^2}$$

The new first-class constraints ψ_i are abelian, $\{\psi_i\,,\psi_j\} = 0$, and have also zero Poisson brackets with other constraints, while for the second-class constraints $\chi_{k\perp}^a$ non-zero Poisson brackets read

$$\{\chi_{i\perp}^a\,,\chi_{j\perp}^b\} = 2\,\epsilon^{abc}\,A_-^c\,\delta_{ij}\,,$$
$$\{\varphi_a^{(2)}\,,\chi_{k\perp}^b\} = \epsilon^{abc}\,\chi_{k\perp}^c\,.$$

Thus, there are 8 first-class constraints $\varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}$ and 4 second-class constraints $\chi_{k\perp}^a$.

Gauge Transformations and Unconstrained Model

Generator of gauge transformations

$$G = \left(-\dot{\varepsilon}_a^{(2)} + \epsilon_{abc}\varepsilon_b^{(2)} A_+^c + \eta_i A_i^a \right) \phi_a^{(1)} + \eta_i \psi_i + \varepsilon_a^{(2)} \phi_a^{(2)}$$

leads to the unconstrained Hamiltonian (Gerdt, Khvedelidze, Mladenov)

$$H = \frac{g^2}{2} \left(p_1^2 + \frac{p_{\theta_3}^2}{4} \frac{1}{q_1^2} \right)$$

describing conformal mechanics.

Contents

Homogeneous Gröbner Basis

With the grading Γ determined by the weights of the variables:

$$\Gamma(\pi_a^\mu) = 2\,, \qquad \Gamma(A_\mu^a) = 1\,, \qquad a = 1, 2, \ldots, 8\,, \qquad \mu = -\,, 1, 2\,,$$

we have the set of homogeneous polynomials ($k = 1, 2$)

$\Gamma -$ degree	Constraints
2	$\chi_k^a = \pi_a^k - \mathrm{f}^{abc} A_-^b A_k^c$
3	$\varphi_a^{(2)} = \mathrm{f}_{abc} \left(A_-^b \pi_c^- + A_k^b \pi_c^k \right)$
5	$\zeta_i = d_{abc} A_i^a F_{-k}^b F_{-k}^c$

The lexicographical order is

$$\pi_a^- \succ \pi_b^1 \succ \pi_c^2 \succ A_-^a \succ A_1^b \succ A_2^c \qquad a, b, c = 1, 2, \ldots, 8\,,$$

and for variables with the same spatial index μ we choose

$$\pi_a^\mu \succ \pi_b^\mu \succ A_\mu^a \succ A_\mu^b \qquad \text{if} \quad a < b\,.$$

Some Simplifications

To simplify calculations we exclude some numerical coefficients by redefinition of variables

$$A^8_- \to A^8_-/\sqrt{3} \qquad P_8^- \to \sqrt{3}P_8^-$$
$$A^8_i \to A^8_i/\sqrt{3} \qquad P_8^i \to \sqrt{3}P_8^i$$

and multiplying of constraints by appropriate factors

$$\chi^a_k \to 2 \times \chi^a_k \qquad \chi^8_k \to \chi^8_k/\sqrt{3}$$
$$\overset{(2)}{\phi}_a \to 2\times \overset{(2)}{\phi}_a \qquad \overset{(2)}{\phi}_8 \to \overset{(2)}{\phi}_8/\sqrt{3}$$

$$\zeta_i \to 8 \times \zeta_i$$

Computational Steps

With such a choice of grading the constraints χ_k^a and $\varphi^{(2)}$ are the lowest degree homogeneous Gröbner basis elements G_2 and G_3 of the order 2 and 3, respectively. Higher degree elements of the basis are constructed step by step by doing the following manipulations:

(i) formation of $S-$polynomials (G_i, G_j)

(ii) elimination of some superfluous $S-$polynomials according to the Buchberger's criteria

(iii) computation of the normal forms of S-polynomials modulo the lower order elements with respect to the grading chosen.

Results

The results of computation of the Gröbner basis elements of different orders n are shown in the following table where we explicitly indicated only $S-$polynomials with non-vanishing normal form.

G_n	Polynomials #	Constraints and S-polynomials
G_2	16	χ_k^a
G_3	8	$\varphi_a^{(2)}$
G_4	15	$(G_3,\ G_3)$
G_5	14	$\zeta_i,\ (\zeta_i, G_j) \qquad i = 1,2 \quad j = 2,3,4$ $(G_2, G_4), (G_3, G_3), (G_3, G_4), (G_4, G_4)$
G_6	13	$(G_2, G_5), (G_3, G_5), (G_4, G_5), (G_5, G_5)$ $(G_3, G_4), (G_4, G_4)$

Results (cont.)

With another lexicographical order

$$A_1^b \succ A_2^c \succ A_-^a \succ \pi_b^1 \succ \pi_c^2 \succ \pi_a^- \qquad a, b, c = 1, 2, \ldots, 8\,,$$

G_n	Polynomials #	Constraints and S-polynomials
G_2	16	χ_k^a
G_3	72	$(G_2,\ G_2)$
G_4	176	$(G_2,\ G_3),\ (G_3,\ G_3)$
G_5	376	$(G_2, G_4), (G_3, G_3), (G_3, G_4), (G_4, G_4)$

G_3 contains:

$$\psi_i = A_-^a \chi_i^a\,, \qquad i = 1, 2$$

$$A_1^a \chi_1^a, \quad A_2^a \chi_2^a, \quad A_1^a \chi_2^a + A_2^a \chi_1^a\,.$$

ζ_i have other, “more simpler” form ($F^a_{-k} = \mathrm{f}^{abc} A_-^b\ A_k^c$)

$$\zeta_i = \mathrm{d}^{abc} A_i^a F^b_{-k} F^c_{-k} \qquad \to \qquad \zeta_i = \mathrm{d}^{abc} A_i^a \pi_b^k \pi_c^k$$

Results (cont.)

Calculations were performed with *Mathematica* (version 5.0) on the machine 2xOpteron-242 (1.6 Ghz) with 6Gb of RAM and have take about a month.

For the structure group $SU(2)$ we used the built-in-function `GroebnerBasis` with monomial order `DegreeReverseLexicographic`

$$\{\pi_1^1, \pi_1^2, \pi_2^1, \pi_2^2, \pi_3^1, \pi_3^2, \pi_1^-, \pi_2^-, \pi_3^-, A_1^1, A_2^1, A_1^2, A_2^2, A_1^3, A_2^3, A_-^1, A_-^2, A_-^3\}\,.$$

In this case the construction of the complete homogeneous Gröbner basis of 64 elements takes about 60 seconds.

Conclusions

- Dirac's Hamiltonian formalism for degenerate mechanical systems with polynomial Lagrangians admit full algorithmisation of the following steps: computation and separation of the complete set of constraints and construction of the gauge symmetry generator.
- Gröbner or involutive bases form the fundamentals of the algorithmisation since these bases allow to work algorithmically modulo constraints.
- Algorithmisation of determinating the unconstrained observables and of the Hamiltonian reduction to these observables still remain to be done.
- For the $SU(2)$ Yang-Mills light-cone mechanics the Hamiltonian reduction has been performed.
- For the $SU(3)$ Yang-Mills light-cone mechanics due to the large number of variables and constraints the special homogeneous Gröbner basis has been constructed in the Mathematica codes. This allowed us to compute the complete set of constraints.