Physarum Can Compute Shortest Paths: Convergence Proofs and Complexity Bounds

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Abstract. *Physarum polycephalum* is a slime mold that is apparently able to solve shortest path problems. A mathematical model for the slime's behavior in the form of a coupled system of differential equations was proposed by Tero, Kobayashi and Nakagaki [TKN07]. We prove that a discretization of the model (Euler integration) computes a $(1+\epsilon)$-approximation of the shortest path in $O(mL(\log n + \log L)/\epsilon^3)$ iterations, with arithmetic on numbers of $O(\log(nL/\epsilon))$ bits; here, n and m are the number of nodes and edges of the graph, respectively, and L is the largest length of an edge. We also obtain two results for a directed Physarum model proposed by Ito et al. [IJNT11]: convergence in the general, nonuniform case and convergence and complexity bounds for the discretization of the uniform case.

1 Introduction

Physarum polycephalum is a slime mold [BD97] that is apparently able to solve shortest path problems. In [NYT00], Nakagaki, Yamada, and Tóth report on the following experiment (see Figure 1): They built a maze, that was later covered with pieces of Physarum (the slime can be cut into pieces that will merge if brought into each other's vicinity), and then fed the slime with oatmeal at two locations. After a few hours, the slime retracted to the shortest path connecting the food sources in the maze. The experiment was repeated with different mazes; in all experiments, Physarum retracted to the shortest path. Tero, Kobayashi and Nakagaki [TKN07] propose a mathematical model for the behavior of the mold. Physarum is modeled as a tube network traversed by liquid flow, with the flow satisfying the standard Poiseuille assumption from fluid mechanics. In the following, we use terminology from the theory of electrical networks, relying on the fact that equations for electrical flow and Poiseuille flow are the same [Kir10].

In particular, let G be an undirected graph[1] with node set N, edge set E, length labels $l \in \mathbb{R}^E_{++}$ [2]and two distinguished nodes $s_0, s_1 \in N$. In our discussion,

[1] One can easily generalize the model and extend our results to multigraphs at the expense of heavier notation. Details will appear in the full version of the paper.

[2] We let $\mathbb{R}^A$, $\mathbb{R}^A_+$ and $\mathbb{R}^A_{++}$ denote the set of real, nonnegative real, and positive real vectors (respectively) whose components are indexed by A.

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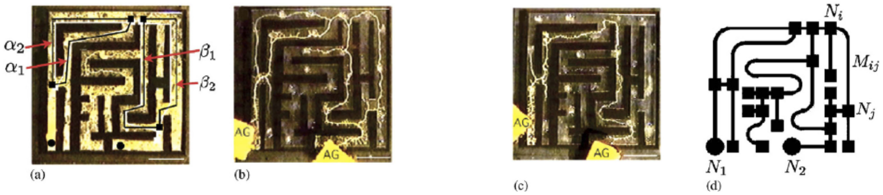

Fig. 1. The experiment in [NYT00] (reprinted from there): (a) shows the maze uniformly covered by Physarum; the yellow color indicates the presence of Physarum. Food (oatmeal) is provided at the locations labelled AG. After a while, the mold retracts to the shortest path connecting the food sources as shown in (b) and (c). (d) shows the underlying abstract graph. The video [You] shows the experiment.

$x \in \mathbb{R}_+^E$ will be a state vector representing the diameters of the tubular channels of the Physarum (edges of the graph). The value x_e is called the *capacity* of edge e. The nodes s_0 and s_1 represent the location of two food sources. Physarum's dynamical system is described by the system of differential equations [TKN07]

$$\dot{x} = |q(x,l)| - x. \tag{1}$$

Equation (1) is called the *evolution equation*, as it determines the dynamics of the system over time. It is a compact representation of a system of ordinary differential equations, one for every edge of the graph; the absolute value operator $|\cdot|$ is applied componentwise. The vector $q \in \mathbb{R}^E$, known as the *current flow*, is determined by the capacities and lengths of the edges, as follows (see Section 2 for the precise definitions). Force one unit of current from the source to the sink in an electrical network, where the resistance r_e of edge e is given by $r_e \stackrel{\text{def}}{=} l_e/x_e$, and call q_e the resulting current across edge e. In [BMV12, Bon13], it was shown that the dynamics (1) converges to the shortest source-sink path in the following sense: the potential difference between source and sink converges to the length of the shortest source-sink path, the capacities of the edges on the shortest source-sink path[3] converge to one, and the capacities of all other edges converge to zero.

Our first contribution relies on a numerical approximation of (1), as given by Euler's method [SM03],

$$\Delta x = h \cdot (|q(x,l)| - x)\,, \tag{2}$$

or, making the dependency on time explicit,

$$x(t+1) - x(t) = h \cdot (|q(x(t),l)| - x(t))\,, \tag{3}$$

where $h \in (0,1)$ is the step size of the discretization. We prove that the dynamics (3) converges to the shortest source-sink path. More precisely, let opt be the length of the shortest path, n and m be the number of nodes and edges of the

[3] We assume uniqueness of the shortest path for simplicity of exposition.

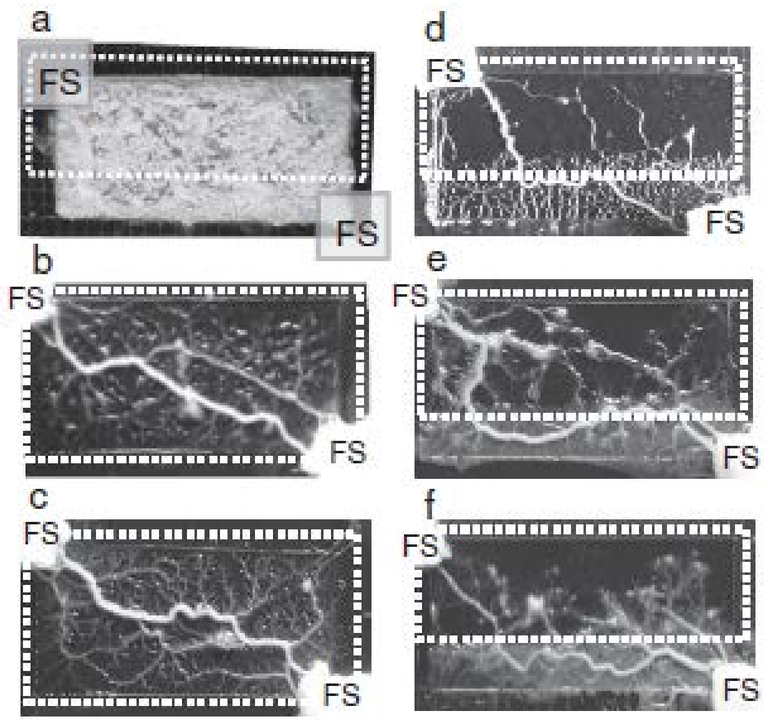

Fig. 2. Photographs of the connecting paths between two food sources (FS). (a) The rectangular sheet-like morphology of the organism immediately before the presentation of two FS and illumination of the region indicated by the dashed white lines. (b),(c) Examples of connecting paths in the control experiment in which the field was uniformly illuminated. A thick tube was formed in a straight line (with some deviations) between the FS. (d)-(f) Typical connecting paths in a nonuniformly illuminated field (95 K lx). Path length was reduced in the illuminated field, although the total path length increased. Note that fluctuations in the path are exhibited from experiment to experiment. (Figure and caption reprinted from [NIU$^+$07, Figure 2].)

graph, and L be the largest length of an edge. We show that, *for* $\epsilon \in (0, 1/300)$ *and for* $h = \epsilon/mL$, *the discretized model yields a solution of value at most* $(1 + O(\epsilon))\mathrm{opt}$ *in* $O(mL(\log n + \log L)/\epsilon^3)$ *steps, even when* $O(\log(nL/\epsilon))$*-bit number arithmetic is used.* For bounded L, the time bound is therefore polynomial in the size of the input data.

Our second contribution was inspired by the following experiment of Nakagaki et al., reported in [NIU$^+$07] (see also Figure 2). They cover a rectangular plate with Physarum and feed it at opposite corners of the plate. Two-thirds of the plate are put under a bright light, and one-third is kept in the dark. Under uniform lighting conditions, Physarum would retract to a straight-line path connecting the food sources [NYT00]. However, Physarum does not like light and therefore forms a path with one kink connecting the food sources. The path is such that the part under light is shorter than in a straight-line connection. In the theory section of [NYT00], a reactivity parameter $a_e > 0$ is introduced into (1):

$$\dot{x}_e(t) = |q_e(x,l)| - a_e x_e(t). \tag{4}$$

Note that if, for example, $q_e(x,l) = 0$, the capacity of edge e decreases with a rate that depends on a_e. To model the experiment, $a_e = 1$ for edges in the dark part of the plate, and $a_e = C > 1$ for the edges in the lighted area, where C is a constant. The authors of [NIU+07] report that in computer simulations, the dynamics (4) converges to the shortest source-sink path with respect to the modified length function $a_e l_e$. A proof of convergence is currently only available for the uniform case $a_e = 1$ for all e, see [BMV12, Bon13].

A directed version of model (4) was proposed in [IJNT11]. The graph $G = (N, E)$ is now a directed graph. For a state vector $x(t)$, the flows are defined as above. A flow $q_e(x,l)$ is positive if it flows in the direction of e and is negative otherwise. The dynamics becomes

$$\dot{x}_e(t) = q_e(x,l) - a_e x_e(t). \tag{5}$$

Although this model apparently has no physical counterpart, it has the advantage of allowing one to treat directed graphs. Ito et al. [IJNT11] prove convergence to the shortest source-sink path in the uniform case ($a_e = 1$ for all e). In fact, they show convergence for a somewhat more general problem, the transportation problem, as does [BMV12] for the undirected model.

We show that the dynamics (5) *converges to the shortest directed source-sink path under the modified length function* $a_e l_e$. This generalizes the convergence result of [IJNT11] from the uniform ($a_e = 1$ for all e) to the nonuniform case, albeit only for the shortest path problem. Our proof combines arguments from [MO07, MO08, IJNT11, BMV12, Bon13] and we believe it is simpler than the one in [IJNT11]. Moreover, for the uniform case (that is, $a_e = 1$ for all e), we can prove convergence for the discretized model

$$x_e(t+1) = x_e(t) + h(q_e(x,l) - x_e(t)), \tag{6}$$

where $h \leq 1/(n(4nm^2LX_0^2)^2)$ is the step size; here, X_0 is the maximum between the largest capacity and the inverse of the smallest capacity at time zero. In particular, let P^* be the shortest directed source-sink path and let $\epsilon \in (0,1)$ be arbitrary: *we show* $x_e(t) \geq 1 - 2\epsilon$ *for* $e \in P^*$ *and* $x_e(t) \leq \epsilon$ *for* $e \notin P^*$, *whenever* $t \geq \frac{4nL}{h}\left(3\ln X_0 + 2\ln\frac{2m}{\epsilon}\right)$.

Outline of the paper. The remainder of the paper is structured as follows. In Section 2 we give basic definitions and properties. In Section 3 we study the discrete dynamics (3). Section 4 concerns the directed models (5) and (6). We close with some concluding remarks in Section 5.

2 Electrical Networks

Without loss of generality, assume that $N = \{1, 2, \ldots, n\}$, $E = \{1, 2, \ldots, m\}$ and assume an arbitrary orientation of the edges.[4] Let $A = (a_{ve})_{v \in N, e \in E}$ be the incidence matrix of G under this orientation, that is, $a_{ve} = +1$ if v is the tail of e, $a_{ve} = -1$ if v is the head of e, and $a_{ve} = 0$, otherwise. Then q is defined as the unit-value flow from s_0 to s_1 of minimum energy, that is, as the unique optimal solution to the following continuous quadratic optimization problem, related to *Thomson's principle* from physics [Bol98, Theorem IX.2]:

$$\min \; q^T Rq \quad \text{such that} \quad Aq = b. \tag{7}$$

Here, $R \stackrel{\text{def}}{=} \operatorname{diag}(l/x) \in \mathbb{R}^{E \times E}$ is the diagonal matrix with value $r_e \stackrel{\text{def}}{=} l_e/x_e$ for the e-th element of the main diagonal, and $b \in \mathbb{R}^N$ is the vector defined by $b_v = +1$ if $v = s_0$, $b_v = -1$ if $v = s_1$, and $b_v = 0$, otherwise. The value r_e is called the *resistance* of edge e. Node s_0 is called the *source*, node s_1 the *sink*. The quantity $\eta \stackrel{\text{def}}{=} q^T Rq$ is the *energy*; the quantity $b_{s_0} = 1$ is the *value* of the flow q. The optimality conditions for (7) imply that there exist values $p_1, \ldots, p_n \in \mathbb{R}$ (*potentials*) that satisfy *Ohm's law* [Bol98, Section II.1]:

$$q_e = (p_u - p_v)/r_e, \qquad \text{whenever edge } e \text{ is oriented from } u \text{ to } v. \tag{8}$$

By the *conservation of energy* principle, the total energy equals the difference between the source and sink potentials, times the value of the flow [Bol98, Corollary IX.4]:

$$\eta = (p_{s_0} - p_{s_1}) b_{s_0} = p_{s_0} - p_{s_1}. \tag{9}$$

3 Convergence of the Undirected Physarum Model

In this section we characterize Physarum's convergence properties in the undirected model, as given by equation (3):

$$x(t+1) = x(t) + h \cdot (|q(x(t), l)| - x(t)).$$

Assumptions on the input data: We assume that the length labels l and the initial conditions $x(0)$ satisfy the following:

a. each s_0-s_1 path in G has a distinct overall length; in particular, there is a unique shortest s_0-s_1 path;
b. all capacities are initialized to one:

$$x(0) = 1; \tag{10}$$

[4] In the directed model discussed in Section 4, this orientation is simply the one given by the directed graph.

c. the initially minimum capacity cut is the source cut, and it has unit capacity:

$$\mathbf{1}_S^T \cdot x(0) \geq \mathbf{1}_0^T \cdot x(0) = 1, \qquad \text{for any } s_0\text{-}s_1 \text{ cut } S, \tag{11}$$

where $\mathbf{1}_S$ is the characteristic vector of the set of edges in the cut S, and $\mathbf{1}_0$ is the characteristic vector of the set of edges incident to the source. Notice that this can be achieved even when s_0 has not degree 1, by connecting a new source s_0' to s_0 via a length 1, capacity 1 edge.

d. every edge has length at least 1.

Basic properties: The first property we show is that the set of fractional s_0-s_1 paths is an invariant for the dynamics.

Lemma 1. *Let $x = x(t)$ be the solution of* (3) *under the initial conditions $x(0) = 1$. The following properties hold at any time $t \geq 0$: (a) $x > 0$, (b) $\mathbf{1}_S^T \cdot x \geq \mathbf{1}_0^T \cdot x = 1$, and (c) $x \leq 1$.*

Proof. (a.) Let $e \in E$ be any edge. Since $|q_e| \geq 0$, by the evolution equation (3) we have $\Delta x_e(t) = h(|q_e| - x_e(t)) \geq -hx_e(t)$. Therefore, by induction, $x_e(t+1) \geq x_e(t) - hx_e(t) = (1-h)x_e(t) > 0$ as long as $h < 1$.

(b.) We use induction. The property is true for $x(0)$ by the assumptions on the input data. Then, using (3), induction, and the fact that $\mathbf{1}_S^T \cdot |q| \geq 1$ for any cut S,

$$\mathbf{1}_S^T \cdot x(t+1) = \mathbf{1}_S^T \cdot (x(t) + h(|q| - x(t))) = (1-h)\mathbf{1}_S^T \cdot x(t) + h\mathbf{1}_S^T \cdot |q| \geq 1 - h + h = 1.$$

The fact that $\mathbf{1}_0^T \cdot x = 1$ can be shown similarly.

(c.) Easy induction, along the same lines as the proof of (a.). □

An *equilibrium point* of (3) is a vector $x \in \mathbb{R}_+^E$ such that $\Delta x = 0$. Our assumptions imply that there are a finite number of equilibrium points: each equilibrium corresponds to an s_0-s_1 path of the network, and vice versa.

Lemma 2. *If $x = \mathbf{1}_P$ for some s_0-s_1 path P, then x is an equilibrium point. Conversely, if x is an equilibrium point, then $x = \mathbf{1}_P$ for some s_0-s_1 path P.*

Proof. The proof proceeds along the same lines as for the continuous case, see [Bon13, Lemma 2.3]. □

Convergence: Recall that, by (9),

$$\eta = \sum_{e \in E} r_e q_e^2 = q^T R q = p_{s_0} - p_{s_1}, \tag{12}$$

and let

$$V \stackrel{\text{def}}{=} l^T x = \sum_{e \in E} l_e x_e = \sum_{e \in E} r_e x_e^2 = x^T R x. \tag{13}$$

Here η is the energy dissipated by the system, as well as the potential difference between source and sink. Notice that the quantity V can be interpreted as the "infrastructural cost" of the system; in other terms, it is the cost that would be incurred if every link were traversed by a flow equal to its current capacity. While η may decrease or increase during the evolution of the system, we will show that $\eta \leq V$ and that V is always decreasing, except on equilibrium points.

Lemma 3. *$\eta \leq V$.*

Proof. To see the inequality, consider any flow f of maximum value subject to the constraint that $0 \leq f \leq x$. The minimum capacity of a source-sink cut is 1 at any time, by Lemma 1(b). Therefore, by the Max Flow-Min Cut Theorem, the value of the flow f must be 1. Then by (7),

$$\eta = q^T Rq \leq f^T Rf \leq x^T Rx = V. \qquad \square$$

Lemma 4. *V is a Lyapunov function for* (3)*; in other words, it is continuous and satisfies (i) $V \geq 0$ and (ii) $\Delta V \leq 0$. Moreover, $\Delta V = 0$ if and only if $\Delta x = 0$.*

Proof. V is continuous and nonnegative by construction. Moreover,

$$\begin{aligned}
\Delta V/h = l^T \Delta x/h = l^T(|q| - x) & \quad \text{by (3)}, \\
= x^T R\,|q| - x^T Rx & \quad \text{by } l = Rx, \\
= (x^T R^{1/2}) \cdot (R^{1/2}\,|q|) - x^T Rx & \\
\leq (x^T Rx)^{1/2} \cdot (q^T Rq)^{1/2} - x^T Rx & \quad \text{by Cauchy-Schwarz [Ste04]}, \\
= (\eta V)^{1/2} - V, & \\
\leq V - V & \quad \text{by Lemma 3.} \\
= 0. &
\end{aligned}$$

Observe that $\Delta V = 0$ is possible only when equality holds in the Cauchy-Schwarz inequality. This, in turn, implies that the two vectors $R^{1/2}x$ and $R^{1/2}\,|q|$ are parallel, that is, $|q| = \lambda x$ for some $\lambda \in \mathbb{R}$. However, by Lemma 1(b), the capacity of the source cut is 1 and, by (7), the sum of the currents across the source cut is 1. Therefore, $\lambda = 1$ and $\Delta x = h(|q| - x) = 0$. $\square$

Corollary 1. *As $t \to \infty$, $x(t)$ approaches an equilibrium point of* (3)*, and $\eta(t)$ approaches the length of the corresponding s_0-s_1 path.*

Proof. The existence of a Lyapunov function V implies [LaS76, Theorem 6.3] that $x(t)$ approaches the set $\{x \in \mathbb{R}_+^E : \Delta V = 0\}$, which by Lemma 4 is the same as the set $\{x \in \mathbb{R}_+^E : \Delta x = 0\}$. Since this set consists of isolated points (Lemma 2), $x(t)$ must approach one of those points, say the point $\mathbf{1}_P$ for some s_0-s_1 path P. When $x = \mathbf{1}_P$, one has $\eta = V = \mathbf{1}_P^T \cdot l$. $\square$

Convergence to an approximate shortest path and convergence time: We will track the convergence process via three main quantities: two of these, η and V, have already been introduced. The third one is defined as

$$W \stackrel{\text{def}}{=} \sum_{e \in P^*} l_e \ln x_e,$$

where P^* is the shortest path. Recall that opt denotes the length of P^*. Observe that $W(t) \le 0$ for all t (due to Lemma 1(c)) and $W(0) = 0$ due to the choice of initial conditions. Also observe that $V(0) = l^T \cdot x(0) = \sum_{e \in E} l_e \le mL$, where m is the number of edges of the graph and L is the length of the longest edge.

For a fixed $\epsilon \in (0, 1/300)$, we set $h = \epsilon/mL$. We will bound the number of steps before V falls below $(1+3\epsilon)^3\text{opt} < (1+10\epsilon)\text{opt}$.

Definition 1. *We call a V-*step *any time step t such that $\eta(t) \le (1+3\epsilon)\text{opt}$ and $V(t) > (1+3\epsilon)^3\text{opt}$. We call a W-*step *any time step t such that $\eta(t) > (1+3\epsilon)\text{opt}$ and $V(t) > (1+3\epsilon)^3\text{opt}$.*

Lemma 5. *The number k_V of V-steps is at most $O((\log n + \log L)/(h\epsilon))$.*

Proof. For any V-step t we have, by the proof of Lemma 4 and the assumptions on η and V,

$$\begin{aligned} \Delta V &\le h((\eta V)^{1/2} - V) = hV((\eta/V)^{1/2} - 1) \\ &\le hV(1/(1+3\epsilon) - 1) \le -hV(3\epsilon/(1+\epsilon)) \le -h\epsilon V \end{aligned}$$

so that $V(t+1) \le (1 - h\epsilon)V(t)$. In other words, V decreases by at least an $h\epsilon$ factor at each V-step. Moreover, V is nonincreasing at every step of the whole process, and after it gets below $(1+3\epsilon)^3\text{opt}$ there are no more V-steps. Therefore, the number of V-steps, k_V, is at most $\log_{1/(1-h\epsilon)}(V(0)/\text{opt}) \le (\ln V(0))/(h\epsilon) = O(\log(mL)/(h\epsilon))$ (we used the assumption that $\text{opt} \ge 1$). □

Lemma 6. *At every W-step, W increases by at least* $\text{opt} \cdot h\epsilon/2$.

Proof. Let P^* be the shortest path, so that $1_{P^*}^T \cdot l = \text{opt}$. For a W-step t, we have

$$W(t+1) - W(t) = \sum_{e \in P^*} l_e \ln \frac{x_e(t+1)}{x_e(t)} = \sum_{e \in P^*} l_e \ln \left(1 + h\left(\frac{|p_u - p_v|}{l_e} - 1\right)\right),$$

where u, v are the endpoints of edge e. Using the bound $\ln(1+z) \ge z/(1+z)$, which is valid for any $z > -1$ (recall that $h < 1$), we obtain

$$\begin{aligned} W(t+1) - W(t) &\ge \sum_{e \in P^*} l_e \frac{h\left(\frac{|p_u - p_v|}{l_e} - 1\right)}{1 + h\left(\frac{|p_u - p_v|}{l_e} - 1\right)} = \sum_{e \in P^*} \frac{h\left(|p_u - p_v| - l_e\right)}{1 + h\left(\frac{|p_u - p_v|}{l_e} - 1\right)} \\ &= h \cdot \left(\sum_{e \in P^*} \frac{|p_u - p_v|}{1 + h\left(\frac{|p_u - p_v|}{l_e} - 1\right)} - \sum_{e \in P^*} \frac{l_e}{1 + h\left(\frac{|p_u - p_v|}{l_e} - 1\right)}\right) \\ &\ge h \cdot \left(\sum_{e \in P^*} \frac{|p_u - p_v|}{1 + h\eta} - \sum_{e \in P^*} \frac{l_e}{1 - h}\right), \end{aligned}$$

where we used $\frac{|p_u - p_v|}{l_e} - 1 < \eta$ (we are using the assumption $l_e \geq 1$ for all e). Since $\sum_{e \in P^*} |p_u - p_v| \geq \eta$ and $\eta \leq V \leq mL$, we obtain further

$$W(t+1) - W(t) \geq h\left(\frac{\eta}{1+hmL} - \frac{\text{opt}}{1-h}\right) = h\left(\frac{(1-h)\eta - (1+hmL)\text{opt}}{(1-h)(1+hmL)}\right)$$
$$\geq \text{opt} \cdot h\left(\frac{(1-\epsilon)(1+3\epsilon) - (1+\epsilon)}{(1-h)(1+\epsilon)}\right) > \text{opt} \cdot h\frac{\epsilon - 3\epsilon^2}{1+\epsilon} \geq \frac{\text{opt} \cdot h\epsilon}{2},$$

where the third inequality follows since $\epsilon = hmL$ by definition of h and since $h = \epsilon/(mL) \leq \epsilon$ (note that $mL \geq 1$ from the definition of L). The fourth inequality follows from simple calculus, while the fifth follows since $(1-3\epsilon^2)/(1+\epsilon) \geq 1/2$, whenever $\epsilon \leq 1/3$. □

Lemma 7. *At every V-step, W decreases by at most* $2\text{opt} \cdot h$.

Proof. Trivially, $x_e(t+1) \geq (1-h)x_e(t)$, hence $\ln x_e(t+1) \geq \ln x_e(t) - \ln(1/(1-h)) \geq \ln x_e(t) - 2h$ (since $h < 1/2$). The claim follows from the definition of W. □

Lemma 8. *The number k_W of W-steps is at most $4k_V/\epsilon = O(mL(\log n + \log L)/\epsilon^3)$.*

Proof. At every W-step, W increases by at least $\text{opt} \cdot h\epsilon/2$. But W is always bounded above by 0, is decreased by at most $2\text{opt} \cdot h \cdot k_V$, and starts with $W(0) = 0$. The claim follows. □

Theorem 1. *After at most $O(mL(\log n + \log L)/\epsilon^3)$ steps, V decreases below* $(1 + 10\epsilon)\text{opt}$.

Proof. Until the time that V gets below $(1 + 3\epsilon)^3\text{opt} \leq (1 + 10\epsilon)\text{opt}$, every step is either a V-step or a W-step, of which there can be at most $k_V + k_W = O(mL(\log n + \log L)/\epsilon^3)$ in total. □

Approximate Computation. Real arithmetic is not needed for the results of the preceding section; in fact, arithmetic with $O(\log(nL/\epsilon))$ bits suffices. The proof that approximate arithmetic suffices mimics the proof in the preceding section; details are deferred to a full version of the paper.

4 Convergence of the Directed Physarum Model

We characterize Physarum's convergence properties in the directed model. We assume (A1) $x_e(0) > 0$ for all e, (A2) There is a directed path from the source to the sink, (A3) Edge lengths are integral, and (A4) The shortest source-sink path is unique. It is convenient to study the dynamics

$$\dot{x}_e(t) = a_e(q_e(t) - x_e(t)) \tag{14}$$

instead of (5). This is simply a change of variables and a rescaling of the edge lengths. We define several constants: $a_{\min} = \min(1, \min_e a_e)$, $x_{\max}(0) = \max(1, \max_e x_e(0))$, $x_{\min}(0) = \min(1, \min_e x_e(0))$, $X_0 = \max\left(x_{\max}(0), \frac{1}{x_{\min}(0)}\right)$, and $L = \max_e l_e$. P^* denotes the shortest directed source-sink path. We prove:

Theorem 2. *Assume (A1)–(A4) and let* $\epsilon \in (0,1)$ *be arbitrary. If* $t \geq \frac{nL}{a_{\min}} \cdot \left(3 \ln X_0 + 2 \ln \frac{2m}{\epsilon}\right)$*, then* $x_e(t) \geq 1 - 2\epsilon$ *for* $e \in P^*$ *and* $x_e(t) \leq \epsilon$ *for* $e \notin P^*$*.*

Electrical flows are uniquely determined by Kirchhoff's and Ohm's laws. In our setting, the electrical flow $q(t)$ and the vertex potentials $p(t)$ are functions of time. For an edge $e = (u, v)$, let $\eta_e(t) = p_u(t) - p_v(t)$, and let $\eta(t) = p_{s_0}(t) - p_{s_1}(t)$. We have the following facts: (1) For any directed source-sink path P, $\sum_{e \in P} \eta_e(t) = \eta(t)$. (2) $x_e(t) \leq \max(1, x_e(0)) \leq x_{\max}(0)$ for all t. (3) $x_e(t) > 0$ for all $e \in E$ and all t (the existence of a directed source-sink path is crucial here). (4) $\ln x_e(t) = \ln x_e(0) + a_e \left(\frac{\hat{\eta}_e(t)}{l_e} - 1\right) \cdot t$, where $\hat{\eta}_e(t) = (1/t) \int_0^t \eta_e(s) ds$ is the average potential drop on edge e up to time t. For a directed source-sink path P, let

$$l_P = \sum_{e \in P} l_e \quad \text{and} \quad w_P(t) = \sum_{e \in P} \frac{l_e}{a_e} \ln x_e(t).$$

be its length and its weighted sum of log capacities, respectively. The quantity w_P was introduced in [MO07, MO08], and the following property (15) was derived in these papers.

Lemma 9. *Assume (A1), (A2) and let* P *be any directed source-sink path. Then*

$$\dot{w}_P(t) = \eta(t) - l_P \quad \textit{and} \quad \frac{d}{dt}(w_P(t) - w_{P^*}(t)) = l_{P^*} - l_P. \tag{15}$$

Moreover, $w_P(t) \leq (3nL \ln X_0)/a_{\min} - t$*, if* P *is a non-shortest source-sink path and (A3) holds: For* $\epsilon \in (0,1)$*, let* $t_1 = nL(3 \ln X_0 + \ln(1/\epsilon))/a_{\min}$*. Then* $\min_{e \in P} x_e(t) \leq \epsilon$ *for* $t \geq t_1$*.*

The last claim states that for any non-shortest path P, $\min_{e \in P} x_e(t)$ goes to zero. This is not the same as stating that there is an edge in P whose capacity converges to zero. Such a stronger property will be shown in the proof of the main theorem.

The Convergence Proof: The proof proceeds in two steps. We first show that the vector of edge capacities becomes arbitrarily close to a nonnegative non-circulatory flow and then prove the main theorem. A flow is *nonnegative* if $f_e \geq 0$ for all e, and it is *non-circulatory* if $f_e \leq 0$ for at least one edge e on every directed cycle.

Lemma 10. *Assume (A1) and (A2): For* $t > t_0 \stackrel{\text{def}}{=} (1/a_{\min}) \ln(3mX_0)$*, there is a nonnegative non-circulatory flow* $f(t)$ *with*

$$|f_e(t) - x_e(t)| \leq 5mX_0 \cdot e^{-a_{\min} t}. \tag{16}$$

Proof. We follow the analysis in [IJNT11], taking reactivities into account. □

We are now ready for the proof of the main theorem.

Proof (of Theorem 2). Let $\mathcal{P}$ be the set of non-shortest simple source-sink paths, and let $t > t_0$, where t_0 is defined as in Lemma 10. The nonnegative non-circulatory flow $f(t)$ can be written as a sum of flows along simple directed source-sink paths, i.e.,

$$f(t) = \alpha_{P^*}(t)\mathbf{1}_{P^*} + \sum_{P \in \mathcal{P}} \alpha_P(t)\mathbf{1}_P$$

with nonnegative coefficients α_P. This representation is not unique. However, there is always a representation with at most m nonzero coefficients.[5] For any edge e and any path P with $e \in P$, the flow $f_e(t)$ is at least $\alpha_P(t)$.

Let $\epsilon \in (0, 1)$ be arbitrary. For

$$t \geq \frac{1}{a_{\min}} \max\left(\ln \frac{10m^2 X_0}{\epsilon}, nL\left(3\ln X_0 + \ln \frac{2m}{\epsilon}\right)\right),$$

we have $|f_e(t) - x_e(t)| \leq \epsilon/(2m)$ for all e (Lemma 10) and $\min_{e \in P} x_e(t) \leq \epsilon/(2m)$ for every non-shortest path P (Lemma 9). Thus, every non-shortest path contains an edge e with $f_e(t) \leq \epsilon/m$. Thus, $\alpha_P(t) \leq \epsilon/m$ for all non-shortest paths P, and hence,

$$x_e(t) \leq m\epsilon/m \leq \epsilon \quad \text{for all } e \notin P^*.$$

The value of the flow f is one. The total flow along the non-shortest paths is at most ϵ. Thus the flow along P^* is at least $1 - \epsilon$. Hence $x_e(t) \geq 1 - \epsilon - \epsilon/(2m) \geq 1 - 2\epsilon$ for all $e \in P^*$. Finally, $\ln \frac{10m^2 X_0}{\epsilon} \leq nL(3\ln X_0 + 2\ln \frac{2m}{\epsilon})$. □

Discretization. We study the discretization of the system of differential equations (14). We proceed in discrete time steps $t = 0, 1, 2, \ldots$ and define the dynamics

$$x_e(t+1) = x_e(t) + h a_e(q_e(t) - x_e(t)), \tag{17}$$

where h is the step size. We will need the following additional assumptions: (A5) $a_e = 1$ for all e, and (A6) there is an edge $e_0 = (s_0, s_1)$ of length nL and initial capacity 0. Observe that the existence of this edge does not change the shortest directed source-sink path. Our main theorem becomes the following; the proof structure for the discrete case is similar to the one for the continuous case.

Theorem 3. *Assume (A1)–(A6) and* $h \leq \frac{1}{24 \cdot n(4nm^2L(X_0)^2)^2}$. *Let* $\epsilon \in (0, 1)$ *be arbitrary. For*

$$t \geq \frac{4nL}{h}\left(3\ln X_0 + 2\ln \frac{2m}{\epsilon}\right),$$

$x_e(t) \geq 1 - 2\epsilon$ *for* $e \in P^*$ *and* $x_e(t) \leq \epsilon$ *for* $e \notin P^*$.

[5] Let $\alpha_{P^*}(t)$ be the minimum value of $f_e(t)$ for $e \in P^*$. Subtract $\alpha_{P^*}(t)\mathbf{1}_{P^*}$ from $f(t)$. As long as $f(t)$ is not the zero flow, determine a source-sink path P carrying nonzero flow and set $\alpha_P(t)$ to the minimum value of $f_e(t)$ for $e \in P$. Subtract $\alpha_P(t)\mathbf{1}_P$ from $f(t)$.

5 Conclusions and Future Work

We summarize our three main results: the discretization (3) of the undirected Physarum model computes an $(1+\epsilon)$-approximation of the shortest source-sink path in $O(mL(\log n + \log L)/\epsilon^3)$ iterations with arithmetic on numbers of $O(\log(nL/\epsilon))$ bits. The dynamics (5) of the nonuniform directed Physarum model converges to the shortest directed source-sink path under the modified length function $a_e l_e$. Within time $nLa_{\min}^{-1} \cdot \left(3 \ln X_0 + 2 \ln \frac{2m}{\epsilon}\right)$, an ϵ-approximation is reached. For the uniform model ($a_e = 1$), we also prove convergence of the discretization.

There are many open questions: (i) Convergence of the nonuniform undirected model; (ii) Convergence of the discretized nonuniform directed model; (iii) Are our bounds best possible? In particular, can the dependency on L be replaced by a dependency on $\log L$?

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