Seminar: Vibrations and Structure-Borne Sound in Civil Engineering – Theory and Applications

Survey of Wave Types and Characteristics

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Abstract

Mechanical waves are waves which propagate through a material medium (solid, liquid, or gas) at a wave speed which depends on the elastic and inertial properties of that medium. Often there will be energy transferring accompany with wave propogation. In this small report, we would like to introduce three different main wave types which are longitudinal waves, transverse waves and bending waves. The general procedure for our exploration will be to derive the governing differential wave equations with the help of kinematic, material and equilibrium equations for each individual wave type. Then we have a comparison of main characteristics for different wave types, particularly propagation velocity.

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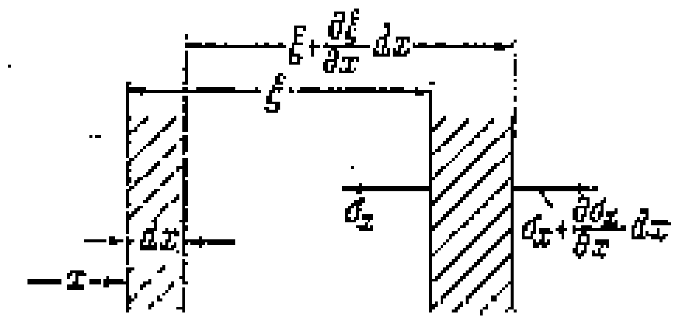
1 Logitudinal Waves
1.1 Pure Longitudinal Waves

Pure longitudinal waves can occur in solids, as well as in liquids and gases. This is defined as waves in which the direction of the particle displacements coincides with the direction of wave propagation. One can readily visualize such waves by studying the motion of two planes which in the undisturbed medium are parallel to each other and perpendicular to the direction of propagation. The kinematic relations are shown in the picture below:



One plane is initially at x and displaced a distance ξ; a second plane, which initially is at a distance dx from the first, is displaced a distance $\xi + \frac{\partial \xi}{\partial x} dx$. The material whose initial length was dx thus experience an extensional strain ε_x in the x-direction, given by $\varepsilon_x = \frac{\partial \xi}{\partial x}$ (1)

Such a strain is associated with a stress, here for the small deformation in relation to structure-borne sound, Hooke's law holds. Then one can write $\sigma_x = D\varepsilon_x$ (2)

Or, by use of Eq. (1), $\sigma_x = D\frac{\partial \xi}{\partial x}$ (2a)

We can see from the picture above that the unbanlanced stress causes the element to accelerate, the corresponding equation of motion maybe written according to Newton's second law as $(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx) - \sigma_x = \rho dx \frac{\partial^2 \xi}{\partial t^2} \quad \rightarrow \quad \frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial^2 \xi}{\partial t^2}$ (3)

It is convenient to describe the kinematics of a sound field in terms of velocity $v_x = \frac{\partial \xi}{\partial t}$ (4)

By introducing the velocity, one may rewrite Eq. (3) as $\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial v_x}{\partial t}$ (5)

One may rewrite Eq. (2a), after differentiation with respect to time, as $\frac{\partial \sigma_x}{\partial t} = D\frac{\partial v_x}{\partial x}$ (6)

We can observe that the relation between the two variables σ_x and v_x is such that the spatial derivative of the one is proportional to the time derivative of the other. So differentiate with respect to t or x and combination of Eqs. (5) and (6) results in the wave eqution,

$$D\frac{\partial^2}{\partial x^2}(\sigma_x, v_x) = \rho\frac{\partial^2}{\partial t^2}(\sigma_x, v_x) \tag{7}$$

We can have a look at the solution of this partial differential equation.

For sinusoidal excitation $P_x = P_0 e^{j\Omega t}$ the corresponding displacement $v_x = v_0 e^{j\Omega(t-x/cL)}$

$$\rightarrow \quad Dv_0 e^{j\Omega(t-x/cL)}\left(\frac{j\Omega}{c_L}\right)^2 = \rho v_0 e^{j\Omega(t-x/cL)}(j\Omega)^2 \quad \rightarrow \quad c_L = \sqrt{\frac{D}{\rho}}$$

We can see that the velocity increases with increasing stiffness and decreases with increasing density.

1.2 Qusai-Longitudinal Waves

The previously discussed pure longitudinal waves can occur only in solids whose dimensions in all directions are much greater than the wave length. We would now have a derivation of the relationship between D (longitudinal stiffness of the material) and E (modulus of elasticity or Young's modulus).

For the fist case, E is defined as the ratio of the stress to strain in the tension direction, which is under the condition of unconstrained cross section (cross-contraction phenomenon occurs).

$$E = \frac{\sigma_x}{\varepsilon_x} \qquad (\sigma_y = \sigma_z = 0) \tag{8}$$

If the lateral contraction is constrained to zero, then there results a three-dimensional instead of a one-dimensional stress condition, because then there are produced the additional normal stresses σ_y and σ_z in the directions normal to the tension direction. These stresses reduce the displacement in the x-direction. Poisson's effects are taken into consideration.

$$\begin{aligned} E\varepsilon_x &= \sigma_x - \mu(\sigma_y + \sigma_z) \\ E\varepsilon_y &= \sigma_y - \mu(\sigma_z + \sigma_x) \\ E\varepsilon_z &= \sigma_z - \mu(\sigma_x + \sigma_y) \end{aligned} \tag{9}$$

For the case where no cross-sectional contraction is permitted, namely for $\varepsilon_y = \varepsilon_z = 0$

One finds by adding the last two of Eqs. (9) that $\sigma_y + \sigma_z = \frac{2\mu}{1-\mu}\sigma_x$, which, after substitution into the first of these equations, leads to $E\varepsilon_x = \sigma_x(1 - \frac{2\mu^2}{1-\mu})$

Thus the longitudinal stiffness which was introduced in Eq. (2), depends on the material parameters E and μ according to the relation:

$$D = \frac{\sigma_x}{\varepsilon_x} = \frac{E}{1 - 2\mu^2/(1-\mu)} = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \qquad (10)$$

Clearly, D is always greater than E.

The wave equation differs from the pure longitudinal wave only in replacement of D by E:

$$\mathrm{E}\frac{\partial^2}{\partial \mathrm{x}^2}(\sigma_\mathrm{x}, \mathrm{v}_\mathrm{x}) = \rho \frac{\partial^2}{\partial \mathrm{t}^2}(\sigma_\mathrm{x}, \mathrm{v}_\mathrm{x}) \qquad (11)$$

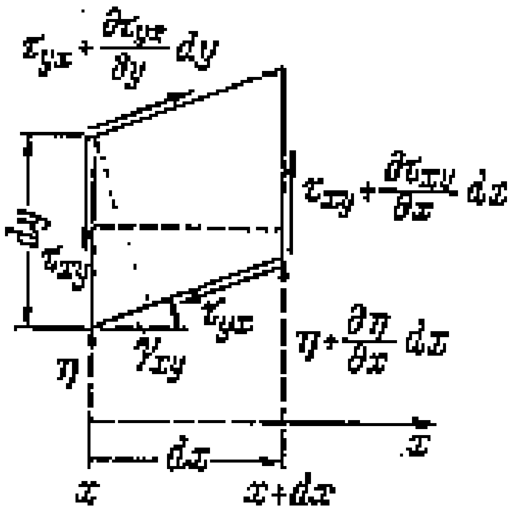
The propagation velocity here is $\mathrm{c}_{\mathrm{LII}} = \sqrt{\frac{\mathrm{E}}{\rho}}$ (12)

This value is smaller than the velocity of pure longitudinal waves. For $\mu = 0.3$, the difference between these two velocities amounts to 16%, which is not entirely negligible. It thus is important to note which longitudinal wave is meant in any given situation.

2 Transverse Waves

2.1 Transverse Plane Waves

Solids do not only resist changes in volume, they also resist changes in shape. This resistance to changes in shape comes about because, unlike a liquid or gas, a solid can support tangential stresses on any cutting plane, even with the material at rest. It is the shear stresses which make it possible for solids to exist in it's own shape. It is also because of shear stresses that transverse plane wave motions can occur in solids bodies, where the direction of propogation is perpendicular to the direction of the displacement. See below the kinematic relations:



Because the transverse displacements of two planes which are a distance dx apart differ by an amount $\frac{\partial \eta}{\partial x} dx$, an element which originally was a rectangle with sides dx and dy is distorted into a parallelogram. The shear angle

$$\gamma_{xy} = \frac{\partial \eta}{\partial x} \qquad (13)$$

The shear deformations always are associated with shear stresses τ_{xy} and τ_{yx}, where the first subscript indicates the axis normal to the plane on which the stress acts, and the second indicates the direction of the stress. Moment equlibrium of the element requires that the shear stresses on two perpendicular plane must be of equal magnitude. These stresses are proportional to the strain γ_{xy} they produce, so that $\tau_{xy} = \tau_{yx} = G\gamma_{xy}$ (14)

With the aid of Eq. (13), $\tau_{xy} = G\dfrac{\partial \eta}{\partial x}$ (14a)

The constant of proportionality G is known as the shear modulus which will be derived later.

The velocity in the y-direction is associated with displacement as $v_y = \dfrac{\partial \eta}{\partial t}$ (15)

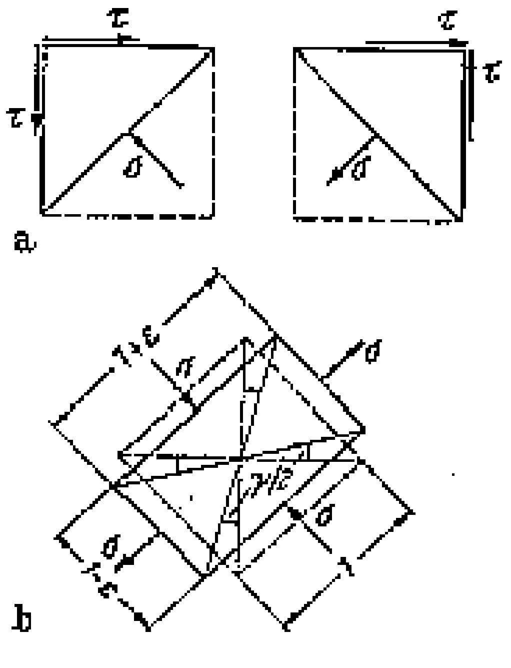
Differentiation of Eq. (14a) with respect to time as $\dfrac{\partial \tau_{xy}}{\partial t} = G\dfrac{\partial v_y}{\partial x}$ (16)

The newton's law relation is $\dfrac{\partial \tau_{xy}}{\partial x} = \rho\dfrac{\partial v_y}{\partial t}$ (17)

This procedure can be totally comparable with the procedure in derivation of pure longitudinal waves. Then there yields wave equation $G\dfrac{\partial^2}{\partial x^2}(\tau_{xy}, v_y) = \rho\dfrac{\partial^2}{\partial t^2}(\tau_{xy}, v_y)$ (18)

From which one finds the propogation velocity is given by $c_T = \sqrt{\dfrac{G}{\rho}}$ (19)

G must be related to E by noticing the fact that normal stresses are always associated with shear stresses, and vice versa. We now have a derivation:



Consider, for example, the diagonal planes of a square whose edges are subject only to shear stresses. Equlibrium of the forces demands that there act on the diagonal plane a compressive or a tensile stress. $2\tau\cos 45^o = \dfrac{\sigma}{\cos 45^o} \rightarrow \tau = \sigma$ (20)

From Eq. (9), we can find that $\varepsilon = \dfrac{\sigma(1+\mu)}{E}$

Strains are related to the angle γ as $\frac{1\text{-}\varepsilon}{1+\varepsilon}=\tan(45^o-\frac{\gamma}{2})\approx\frac{1-\gamma/2}{1+\gamma/2}\rightarrow\varepsilon=\frac{\gamma}{2}$

If we combines these equations with $\tau=G\gamma$

Then we obtains the desired relation between G and E: $\mathrm{G}=\frac{\mathrm{E}}{2(1+\mu)}$ (21)

One may observe that the shear modulus is always considerably smaller than the modulus of elasticity E, and thus much smaller than the longitudinal stiffness D. For μ =0.3, the ratios have the values $\mathrm{c_T}/c_L=0.535,\ and\ \mathrm{c_T}/c_{L\Pi}=0.620$.

2.2 Torsional Waves

If a narrow beam is subjected to a torque, suppose the beam axis coincides with the x-axis,

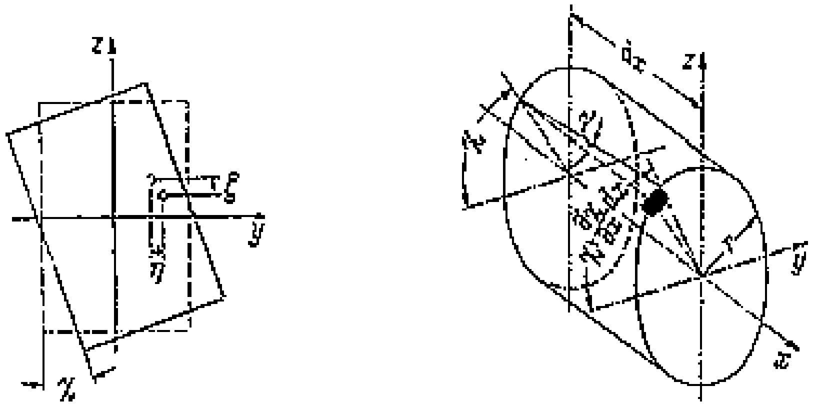


Fig.II/6a. Rotation of cross-section in torsion.

Fig.II/6b. Relation between change in rotation and shear angle.

We can see the relations between the angles that: $\gamma dx=rd\chi\rightarrow\gamma=\mathrm{r}\frac{\mathrm{d}\chi}{\mathrm{dx}}$ where χ represents the angular displacement, in radians, from the equilibrium position.

It is evident to obey that $\tau=G\gamma=G\mathrm{r}\frac{\partial\chi}{\partial\mathrm{x}}$ (22)

Torsional moment is defined as $\mathrm{M_x}=T\frac{\partial\chi}{\partial x}$ (23)

Where T represents the torsional stiffness of a rod with an annular cross-section. If one introduce the time-derivative of the angle of rotation χ, that is the angular velocity about the x axis, $\omega_x=\frac{\partial\chi}{\partial t}$ (24)

Then one may differentiate Eq. (23) with respect to time to obtain $\frac{\partial M_x}{\partial t}=T\frac{\partial\omega_x}{\partial x}$ (25)

Comlemented by the equation $\frac{\partial M_x}{\partial x} = \theta^{'} \frac{\partial \omega_x}{\partial t}$ (26)

We finally reach the wave equation: $T \frac{\partial^2}{\partial x^2}(M_x, \omega_x) = \theta^{'} \frac{\partial^2}{\partial t^2}(M_x, \omega_x)$ (27)

Here $\theta^{'}$ represents the mass moment of inertia.

Propogation velocity in this case is $c_{T\mathrm{I}} = \sqrt{\frac{T}{\theta^{'}}}$ (28)

Now we have a discuss of the value of T and $\theta^{'}$. For rotational symmetric cross-section (like circular or annular cross-section). $T = \frac{\Pi}{2} G(r_\alpha^4 - r_i^4)$ while $\theta^{'} = \frac{\Pi}{2}\rho(r_\alpha^4 - r_i^4)$. Here r_α and r_i represents the outside and inside radii of the cross-section respectively. $c_{T\mathrm{I}} = \sqrt{\frac{G}{\rho}}$ (28a)

If one consider a rectangle cross section instead, we make the cross section narrower in the mean time keep the area constant. We note that c_{TI} ecomes smaller and smaller as the height-to-width ratio becomes larger and larger.

$\frac{h}{b}$	1	1.5	2	3	6	10
$\frac{c_{TI}}{c_T}$	0.92	0.85	0.74	0.56	0.32	0.19

For large values of h/b greater than 6, we can approximately evaluate $\theta^{'} = \frac{\rho(bh^3 + hb^3)}{12} \approx \frac{\rho bh^3}{12}$ and $\mathrm{T} \approx \frac{\rho b^3 h}{3}$. Then $c_{TI} = \sqrt{\frac{T}{\theta^{'}}} = \frac{2b}{h} c_T$ (28b)

We can see from above that the torsional velocity could be far less than transverse velocity.

3 Bending Waves
3.1 Pure Bending Waves

Bending waves are by far the most important for sound radiation because of the rather large lateral deflections associated with them. Bending wave differs largely from both longitudinal waves and transverse waves. It must be represented by 4 field variables instead of 2. Also the boundary conditions are more complex.

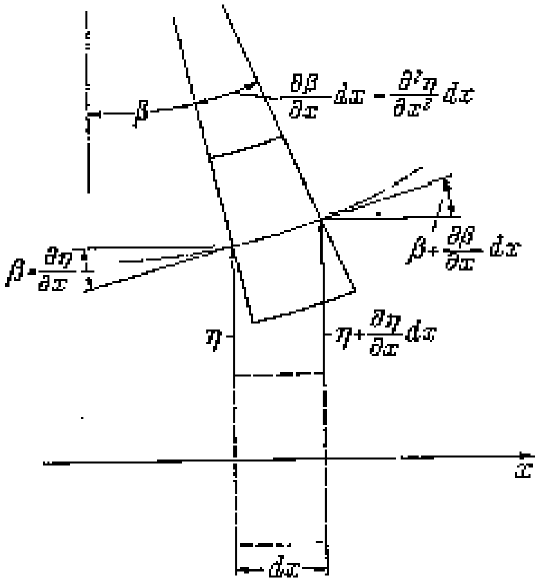


Fig.II/7. Displacements and deformations in bending.

Four filed variables are:

1) transverse velocity v_y
2) angular velocity ω_z
3) bending moment M_Z
4) shear force F_y

The lateral displacement η and the rotation of a cross section through the small angle β are related by the approximate expression $\beta = \frac{\partial\eta}{\partial x}$ (29)

Differentiate with respect to time then leads to $\omega_z = \frac{\partial\beta}{\partial t} = \frac{\partial v_y}{\partial x}$ (30)

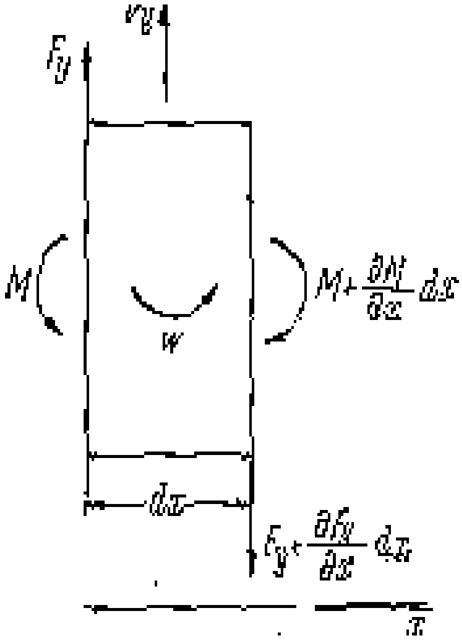
The rate of change of angular velocity with distance is equal to the time-wise rate of change of the curvature, $\frac{\partial\omega_z}{\partial x} = \frac{\partial^2 v_y}{\partial x^2} = \frac{\partial}{\partial t}(\frac{\partial^2\eta}{\partial x^2})$ (31)

As is shown in elmentary strength of materials $M_z = -B\frac{\partial^2\eta}{\partial x^2}$ (32)

Combination of Eqs. (31) and (32) results in $\frac{\partial M_z}{\partial t} = -B\frac{\partial\omega_z}{\partial x}$ (33)

Application of the newton's law in verticle direction , one may write

$$F_y - (F_y + \frac{\partial F_y}{\partial x}dx) = m^{'}dx\frac{\partial v_y}{\partial t} \qquad (34) \text{ which reduces to } -\frac{\partial F_y}{\partial x} = m^{'}\frac{\partial v_y}{\partial t} \qquad (34a)$$



Equlibrium of an element may be written $M_z - (M_z + \frac{\partial M_z}{\partial x}dx) - F_y dx = 0$ (35)

Which can be reduced to $-\frac{\partial M_z}{\partial x} = F_y$ (35a)

From the system of Eqs. (30) (33) (34a) (35a), one obtains the equation for bending waves:

$$-B\frac{\partial^4}{\partial x^4}(v_y, \omega_z, M_z, F_y) = m^{'}\frac{\partial^2}{\partial t^2}(v_y, \omega_z, M_z, F_y) \qquad (36)$$

The propagation velocity here is represented by $c_B = \sqrt{\sqrt{\frac{B}{m^{'}}}}\sqrt{\Omega}$ (37)

Now we give some remark on c_B. It is obvious that the velocity is proportional to the input impulse frequency. In this case, when we have a infinity frequency impulse acting on it, the bending wave propagation velocity could also turn to infinity. This is in contradiction with the conclusion that longitudinal wave propagation velocity must be the largest one. So we need to modify the model to make sense.

3.2 Corrected Bending Waves

The previously discussed pure bending wave model needs to be modified in 2 aspects in order to be apply to a more generous condition.

1) We need to take into consideration of deformations which are caused by shear stresses acting on the cross section. That is the Timoshenko beam theory.

Eq. (29) needs to be written as $\frac{\partial \eta}{\partial x} = \beta + \gamma$ which results in the modification of Eq. (30) into $\frac{\partial v_y}{\partial x} = \omega_z + \frac{\partial \gamma}{\partial t}$. Substitute $\gamma = \frac{\tau}{G} = -\frac{F_y}{GS} = -\frac{F_y}{K}$ into the previously equation will finally generates $\frac{\partial v_y}{\partial x} = \omega_z - \frac{1}{K}\frac{\partial F_y}{\partial t}$ (30a)

2) We need to add the previously omitted rotational inertia term when derive Eq. (35).

$M_z - (M_z + \frac{\partial M_z}{\partial x}dx) - F_y dx = \mathrm{I}\rho dx \frac{\partial \omega_z}{\partial t}$ results in $-\frac{\partial M_z}{\partial x} = F_y + \theta_z^{'} \frac{\partial \omega_z}{\partial t}$ (35b)

From the system of Eqs. (30a) (33) (34a) (35b), one obtains the equation for bending waves:

$$\frac{B}{m^{'}}\frac{\partial^4 v_y}{\partial x^4} + \frac{\partial^2 v_y}{\partial t^2} - [\frac{\theta_z^{'}}{m^{'}} + \frac{B}{K}]\frac{\partial^4 v_y}{\partial x^2 \partial t^2} + \frac{\theta_z^{'}}{K}\frac{\partial^4 v_y}{\partial t^4} = 0 \qquad (38)$$

The first 2 terms correspond to the differential equation for the pure bending waves, the other 3 terms represent the corrections. $\frac{\theta_z^{'}}{m^{'}}$ will occur if rotational inertia is considered. $\frac{B}{K}$ will occur if shear deformation is considered. The last term correspond to higher order correction.

The propagation velocity for corrected bending wave is $c_{B\Pi} = \sqrt{\sqrt{\frac{B}{m^{'}}}\sqrt{\Omega}(1 - 3.6(\frac{h}{\lambda})^2)}$ (39)

References

- Cremer, Lothar, Heckl, Manfred. Structure-borne sound. Structural vibrations and sound radiation at audio frequencies. Published by Springer 1973.