Supersymmetry and Quantum Hall effect in graphene

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1 Introduction

1.1 Topic and message of the report

1.1.1 Graphene as a new material

Apparent importance of materials with specific properties for the development of new or better electronic devices leads to a huge interest in those materials. One of them, thoroughly investigated in recent times, is graphene. Graphene is a single planar sheet of carbon atoms. It can be regarded as a layer of graphite, which is a composite of many parallel carbon sheets. An overview of its properties was given by Anton Lopatin in his talk at MB-JASS 2007.

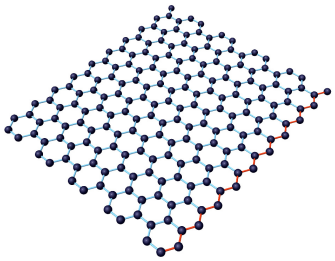

Figure 1: Structure of graphene

Central topic of this work consists of the specific Hamiltonian in graphene and its impact on the Quantum Hall effect.

1.1.2 Graphene Lattice

Graphene has a hexogonal two-dimensional lattice with a diatomic basis. The reciprocal lattice obtained for this type of real space lattice is also two-dimensional and hexagonal.

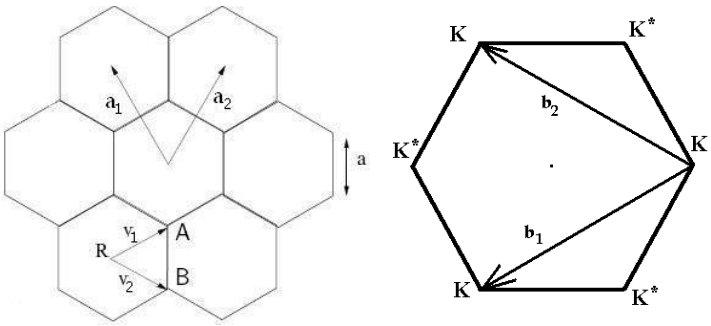

Figure 2: Graphene lattice in real (l.) and reciprocal (r.) space

If $\vec{R}$ is a lattice vector in real space than we can calculate the coordinate vector of 2 atom types contained in the lattice by addition of corresponding displacement vectors in the Wigner-Seitz cell: $\vec{v}_1 + \vec{R} \equiv A$, $\vec{v}_2 + \vec{R} \equiv B$.

1.2 Goal

Similar to the real space lattice the reciprocal lattice of graphene consists of two equal sublattices which are shifted with respect to each other. We call the representatives of these two sublattices K and K*. These two points are independent and symmetric to each other with respect to time-inversion $t \to -t$.

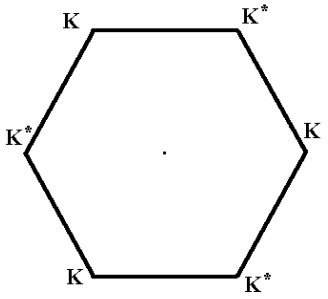

Figure 3: Brillouin zone of graphene

Our first important goal is the effective Hamiltonian around K and K*.

1.3 Motivation of the Hamiltonian

1.3.1 Tight Binding

We write down the wave function of the system corresponding to the tight binding approach:

$$\psi_{\vec{k}}(\vec{r}) = \sum_{\vec{R}} e^{i\vec{k}\vec{R}} \chi(\vec{r} - \vec{R}) \tag{1}$$

$$\psi_{\vec{k}}(\vec{r}) = \alpha \sum_{\vec{R}} e^{i\vec{k}(\vec{R}+\vec{v}_1)} f(\vec{r} - \vec{R} - \vec{v}_1) + \beta \sum_{\vec{R}} e^{i\vec{k}(\vec{R}+\vec{v}_2)} f(\vec{r} - \vec{R} - \vec{v}_2) \tag{2}$$

$$= \sum_{\vec{R}} e^{i\vec{k}\vec{R}} \left[\alpha e^{i\vec{k}\vec{v}_1} f(\vec{r} - \vec{R} - \vec{v}_1) + \beta e^{i\vec{k}\vec{v}_2} f(\vec{r} - \vec{R} - \vec{v}_2) \right] \tag{3}$$

$$=: \sum_{\vec{R}} e^{i\vec{k}\vec{R}} \left(\alpha \varphi^{\alpha}_{\vec{k}}(\vec{r}) + \beta \varphi^{\beta}_{\vec{k}}(\vec{r}) \right) = \alpha \psi^{\alpha}_{\vec{k}}(\vec{r}) + \beta \psi^{\beta}_{\vec{k}}(\vec{r}) \tag{4}$$

1.3.2 Obtaining Hamiltonian

$$\hat{H}\psi_{\vec{k}} = \varepsilon_{\vec{k}} \psi_{\vec{k}}, \quad \psi_{\vec{k}} = \alpha \psi^{\alpha}_{\vec{k}} + \beta \psi^{\beta}_{\vec{k}} \tag{5}$$

$\hat{H}$ acts on vectors in 2-dimensional space spanned by $\binom{1}{0} \equiv \psi^{\alpha}_{\vec{k}} \equiv$ $A-$sublattice and $\binom{0}{1} \equiv \psi^{\beta}_{\vec{k}} \equiv$ $B-$sublattice.

We now calculate the matrix elements, e.g.

$$\hat{H}_{\alpha\beta} = \left\langle \psi^{\alpha}_{\vec{k}} \right| \hat{H} \left| \psi^{\beta}_{\vec{k}} \right\rangle = \sum_{\vec{R},\vec{R}'} e^{-i\vec{k}(\vec{R}+\vec{v}_1-\vec{R}'-\vec{v}_2)} \left\langle f(\vec{r} - \vec{R} - \vec{v}_1) \right| \hat{H} \left| f(\vec{r} - \vec{R}' - \vec{v}_2) \right\rangle = \tag{6}$$

$$= \sum_{\vec{R},\vec{R}'} e^{-i\vec{k}(\vec{R}+\vec{v}_1-\vec{R}'-\vec{v}_2)} \left\langle f(\vec{r} - (\vec{R} + \vec{v}_1 - \vec{R}' - \vec{v}_2)) \right| \hat{H} \left| f(\vec{r}) \right\rangle \approx \tag{7}$$

$$\approx \sum_j t e^{-i\vec{k}\vec{u}_j}, \quad \vec{u}_j = \vec{R} + \vec{v}_1 - \vec{R}' - \vec{v}_2, \ |\vec{u}_j| = |\vec{v}_1 - \vec{v}_2|, \tag{8}$$

where we only consider contributions from neighbouring atoms.

These matrix elements form a 2×2-matrix, which represents the Hamiltonian $\hat{H}$:

$$\hat{H} = \begin{pmatrix} 0 & t\sum_j e^{-i\vec{k}\vec{u}_j} \\ t\sum_j e^{i\vec{k}\vec{u}_j} & 0 \end{pmatrix} \tag{9}$$

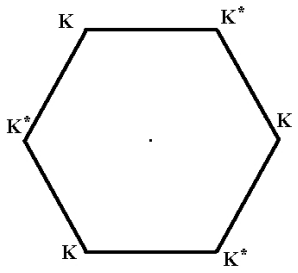

Figure 4: Brillouin zone

Taylor-expansion around $\vec{k}_K$ $\left(\vec{k} = \vec{k}_K + \vec{k'}\right)$ yields

$$\hat{H} = \begin{pmatrix} 0 & (k'_x - ik'_y)v \\ (k'_x + ik'_y)v & 0 \end{pmatrix} \tag{10}$$

More general, using effective mass method:

$$\hat{H} = \begin{pmatrix} 0 & (\hat{p}_x - i\hat{p}_y)v \\ (\hat{p}_x + i\hat{p}_y)v & 0 \end{pmatrix} \tag{11}$$

We obtained the Hamiltonian we will discuss thoroughly from now on.

2 Properties of Hamiltonian

2.1 Without magnetic field

2.1.1 Dirac-Hamiltonian

First recall the effective Hamiltonian around K derived above:

$$\hat{H} = \begin{pmatrix} 0 & (\hat{p}_x - i\hat{p}_y)v \\ (\hat{p}_x + i\hat{p}_y)v & 0 \end{pmatrix} \tag{12}$$

This is the Dirac Hamiltonian of a relativistic particle! The only important difference to the usual Dirac-Hamiltonian is that no spin is involved in the problem. Besides, we do not have the mass term(mc^2) in the Hamiltonian, so we are dealing with extremely relativistic quasiparticles(Weyl-Hamiltonian). Systems of massless particles and quasiparticles(e.g. neutrinos and antineutrinos) can be described in this way.

One should pay attention to the fact, that this Hamiltonian acts around K, but a similar one acts around K*.

2.1.2 Eigenvectors without magnetic field

$$\hat{H} = \begin{pmatrix} 0 & (\hat{p}_x - i\hat{p}_y)v \\ (\hat{p}_x + i\hat{p}_y)v & 0 \end{pmatrix}, \tag{13}$$

which was derived earlier must have eigenvectors

$$\psi_{\vec{k}} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{i\vec{k}\vec{r}},\ \vec{p} = \hbar\vec{k}$$

Using Schroedinger-equation the values α and β can be determined:

$$\psi^{\pm}_{\vec{k}} = \frac{1}{\sqrt{2}} e^{i\vec{k}\vec{r}} \begin{pmatrix} \pm i e^{-i\frac{\theta_k}{2}} \\ e^{i\frac{\theta_k}{2}} \end{pmatrix},\ \varepsilon_{\pm} = \pm\hbar v \left|\vec{k}\right|, \tag{14}$$

where $k_x + ik_y = \left|\vec{k}\right| e^{i\left(\frac{\pi}{2} - \theta_k\right)}$ (15)

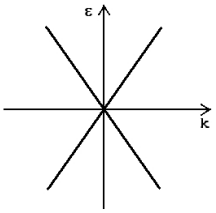

Figure 5: Dispersion relation

2.1.3 Introducing Pauli-matrices

Although no "normal" spin appears in this probem, we use quasi-spin since the Hamiltonian can be described by Pauli-matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{16}$$

$$\text{and their combinations} \tag{17}$$

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma_\pm = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right) \tag{18}$$

Now we can rewrite $\hat{H}$:

$$\hat{H} = \begin{pmatrix} 0 & \hbar(k_x - ik_y)v \\ \hbar(k_x + ik_y)v & 0 \end{pmatrix} = \hbar v \sigma_x k_x + \hbar v \sigma_y k_y = \tag{19}$$

$$= \hbar v \vec{\sigma}_\perp \vec{k}_\perp = \hbar v \left|\vec{k}_\perp\right| \vec{\sigma}_\perp \frac{\vec{k}_\perp}{\left|\vec{k}_\perp\right|} =: \hbar v \left|\vec{k}_\perp\right| \chi_{\vec{k}} \tag{20}$$

$$\vec{\sigma}_\perp = (\sigma_x, \sigma_y),\ \vec{k}_\perp = (k_x, k_y),\ k_\pm = k_x \pm ik_y \tag{21}$$

2.1.4 Chirality operator

Last expression can be written as

$$\hat{H} = \hbar v \left|\vec{k}_\perp\right| \chi_{\vec{k}} \tag{22}$$

using the chirality operator

$$\chi_{\vec{k}} = \vec{\sigma}_\perp \frac{\vec{k}_\perp}{\left|\vec{k}_\perp\right|} \tag{23}$$

Since $\chi_{\vec{k}}$ commutes with Hamiltonian(which means conservation of chirality) chirality is as the projection of quasi-spin in the $\vec{k}-$direction a good quantum number, quasi-spin itself is a bad one. Solutions can be described completely by $\vec{k}$ and $\chi_{\vec{k}}$.

2.2 With magnetic field

2.2.1 Hamiltonian in magnetic field

We now intoruce magnetic field B along the $z-$axis and corresponding vector potential $\vec{A}$:

$$\vec{A} = (0, Bx, 0),\ \vec{B} = \nabla \times \vec{A} \quad \text{(Landau gauge)} \tag{24}$$

This introduction leads to a change of the Hamiltonian according to replacements

$$\hat{\vec{p}} = -i\hbar\nabla \to \hat{\vec{\pi}} = \hat{\vec{p}} - \frac{e}{c}\vec{A}, \quad \hat{H} = v\vec{\sigma}\vec{p} \to v\vec{\sigma}\vec{\pi} = v\vec{\sigma}_\perp \hat{\vec{\pi}}_\perp \tag{25}$$

Here we denote as $\hat{\vec{\pi}}_\perp = (\hat{\pi}_x, \hat{\pi}_y)$ the two-dimensional momentum. We get:

$$\vec{\sigma}_\perp \vec{p}_\perp = \sigma_+ p_- + \sigma_- p_+ \ \Rightarrow\ \vec{\sigma}_\perp \hat{\vec{\pi}}_\perp = \sigma_+ \hat{\pi}_- + \sigma_- \hat{\pi}_+ \tag{26}$$

$$\hat{\pi}_\pm = \hat{p}_\pm \mp i\frac{e}{c}Bx \ \Rightarrow [\hat{\pi}_+, \hat{\pi}_-] = 2\frac{e\hbar}{c}B \tag{27}$$

2.2.2 Analogy with ladder operators

We rewrite this important result in a few different ways

$$[\hat{\pi}_+, \hat{\pi}_-] = 2\frac{e\hbar}{c}B = 2\hbar^2\frac{|eB|}{c\hbar}sgn(eB) = 2\frac{\hbar^2}{l_H^2}sgn(eB) =: \gamma^2 sgn(eB) \tag{28}$$

$$l_H^2 = \frac{c\hbar}{|eB|} \text{ is called magnetic length} \tag{29}$$

We see that the commutator of $\hat{\pi}_+$ and $\hat{\pi}_-$ is a number, whose sign is $sgn(eB)$.

Recall: raising and lowering operators of HO are defined as

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,,\ a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{30}$$

$$\left[a, a^\dagger\right] = 1,\ a^\dagger a = \hat{N} \tag{31}$$

$\hat{\vec{\pi}}_\pm$ can therefore be understood as raising/lowering operators depending on $sgn(eB)$.

3 Supersymmetry methods

3.1 Basics of Supersymmetry

3.1.1 What is supersymmetry?

In the section above we found an analogy of operators in our problem with ladder operators of the harmonic oscillator. More general concept with the same properties is that of supersymmetric operators.

We consider a system with bosons and fermions with variable number of both particle types. Supersymmetry refers to the symmetry between bosons and fermions, i.e. to specific transformations from one type to the other. Supersymmetric states are denoted by $|n_B, n_F\rangle$, where $n_B = 0, 1, 2, \ldots,\ n_F = 0, 1$.

3.1.2 Suppersymmetric operators

Operators of crucial importance are the annihilation(b, f) and creation$(b^\dagger, f^\dagger)$ operators:

$$b\,|n_B, n_F\rangle = \sqrt{n_B}\,|n_B - 1, n_F\rangle\,,\ b^\dagger\,|n_B, n_F\rangle = \sqrt{n_B+1}\,|n_B+1, n_F\rangle \tag{32}$$

$$\left[b, b^\dagger\right] = 1,\ b^\dagger b = \hat{N}_B \tag{33}$$

for bosons and

$$f\,|n_B, n_F\rangle = \sqrt{n_F}\,|n_B, n_F - 1\rangle\,,\ f^\dagger\,|n_B, n_F\rangle = \sqrt{n_F+1}\,|n_B, n_F+1\rangle \tag{34}$$

$$\left\{f, f^\dagger\right\} = ff^\dagger + f^\dagger f = 1,\ f^\dagger f = \hat{N}_F \tag{35}$$

for fermions. Now we define operators which transform bosons to fermions and vice versa:

$$Q_+ = bf^\dagger \ \Rightarrow\ Q_+\,|n_B, n_F\rangle \sim |n_B - 1, n_F + 1\rangle \tag{36}$$

$$Q_- = b^\dagger f \ \Rightarrow\ Q_-\,|n_B, n_F\rangle \sim |n_B + 1, n_F - 1\rangle \tag{37}$$

Their properties can be derived from relations applying for f and b:

$$f^2 = \left(f^\dagger\right)^2 = 0 \ \Rightarrow\ Q_+^2 = Q_-^2 = 0 \tag{38}$$

Besides, let us define their hermitian linear combinations:

$$Q_1 = Q_+ + Q_-,\ Q_2 = -i(Q_+ - Q_-)\ \Rightarrow \{Q_1, Q_2\} = 0 \tag{39}$$

First, one should consider the simplest Hamiltonian consisting of operators defined above:

$$\hat{H} = \{Q_+, Q_-\} = Q_1^2 = Q_2^2 \tag{40}$$

$$[H, Q] = 0,\ \text{where } Q = Q_+, Q_-, Q_1 \text{ or } Q_2 \tag{41}$$

Therefore the values of these operators are conserved in such systems.

3.2 General Hamiltonian

3.2.1 Generalizing Hamiltonian

Back to our Hamiltonian

$$\hat{H} = \begin{pmatrix} 0 & v\pi_- \\ v\pi_+ & 0 \end{pmatrix} \tag{42}$$

We introduce the mass term Δ:

$$\hat{H} = \begin{pmatrix} \Delta & v\pi_- \\ v\pi_+ & -\Delta \end{pmatrix} \tag{43}$$

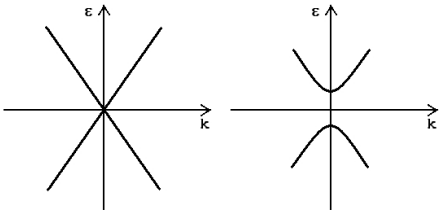

Figure 6: Dispersion relation without(l.) and with(r.) mass

Recall: there are two independent points in momentum space: K and K*. They act in different subspaces, so the total Hamiltonian has the form

$$\begin{pmatrix} \hat{H}_K & 0 \\ 0 & \hat{H}_{K^*} \end{pmatrix} \tag{44}$$

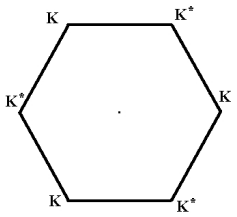

Figure 7: Brillouin zone

3.2.2 Suppersymmetric operators

Summarizing Hamiltonian around K and K^* we get

$$\hat{H} = \begin{pmatrix} \Delta & v\pi_- & & \\ v\pi_+ & -\Delta & & \\ & & \Delta & v\pi_+ \\ & & v\pi_- & -\Delta \end{pmatrix} \tag{45}$$

$$= \begin{pmatrix} \Delta\sigma_z + (\sigma_+\pi_- + \sigma_-\pi_+)v & 0 \\ 0 & \Delta\sigma_z + (\sigma_+\pi_+ + \sigma_-\pi_-)v \end{pmatrix} \tag{46}$$

If we interchange the last two basis vectors we get

$$\hat{H} = \begin{pmatrix} \Delta\sigma_z + (\sigma_+\pi_- + \sigma_-\pi_+)v & 0 \\ 0 & -\Delta\sigma_z + (\sigma_+\pi_- + \sigma_-\pi_+)v \end{pmatrix} \tag{47}$$

We have obtained a four-dimensional Hamiltonian consisting of two two-dimensional ones which differ only by the sign of Δ. To unify these two, we try to calculate the square of one of these two two-dimensional Hamiltonians. We will then see that this square is at the same time the square of the second two-dimensional part.

3.2.3 Squared Hamiltonian

In the last section we derived the total effective Hamiltonian:

$$\hat{H} = \begin{pmatrix} \Delta\sigma_z + (\sigma_+\pi_- + \sigma_-\pi_+)v & 0 \\ 0 & -\Delta\sigma_z + (\sigma_+\pi_- + \sigma_-\pi_+)v \end{pmatrix} \tag{48}$$

Let us look at the square of the Hamiltonian around K:

$$(\Delta\sigma_z + (\sigma_+\pi_- + \sigma_-\pi_+)v)^2 = (\Delta\sigma_z)^2 + ((\sigma_+\pi_- + \sigma_-\pi_+)v)^2 + v\Delta\left\{\sigma_z, \vec{\sigma}_\perp\right\}\hat{\vec{\pi}}_\perp = \tag{49}$$

$$= (\Delta\sigma_z)^2 + ((\sigma_+\pi_- + \sigma_-\pi_+)v)^2 = \tag{50}$$

$$= (-\Delta\sigma_z + (\sigma_+\pi_- + \sigma_-\pi_+)v)^2 \tag{51}$$

$$\text{Since } \{\sigma_z, \vec{\sigma}_\perp\} = 0 \tag{52}$$

$\hat{H}^2 = (\Delta\sigma_z)^2 + ((\sigma_+\pi_- + \sigma_-\pi_+)v)^2$ contains therefore both solutions for K and K*.

Squared Schroedinger-equation has the form

$$\hat{H}^2\psi = \varepsilon^2\psi = (\Delta\sigma_z + v\sigma_\perp\pi_\perp)^2\psi = \Delta^2 + v^2(\sigma_\perp\pi_\perp)^2\psi \tag{53}$$

where we have to tackle the right side yet:

$$(\vec{\sigma}_\perp\hat{\vec{\pi}}_\perp)^2 = \frac{1}{2}\left[\{\hat{\pi_+}, \hat{\pi_-}\} - \gamma^2 sgn(eB)\sigma_z\right] \text{ with} \tag{54}$$

$$\{\hat{\pi_+}, \hat{\pi_-}\}\,|n\rangle = \gamma^2(2n+1)\,|n\rangle \tag{55}$$

which applies always since one of the operators $\hat{\pi_\pm}$ is always the lowering operator, the other is always the raising operator. Only which one is which depends on $sgn(eB)$. Hence,

$$(\sigma_\perp\pi_\perp)^2\,|n\rangle\,|\alpha\rangle = \frac{1}{2}\gamma^2\left[2n + 1 - sgn(eB)\sigma_z\right]|n\rangle\,|\alpha\rangle \tag{56}$$

Landau levels correspond to bosons($|n\rangle$), quasi-spin to fermions($|\alpha\rangle$).

$$\varepsilon^2 = \Delta^2 + \frac{v^2\gamma^2}{2}\left(2n + 1 - sgn(eB)\sigma_z\right) \tag{57}$$

$$\varepsilon^2_{n,\downarrow} = \varepsilon^2_{n-1,\uparrow} = \Delta^2 + v^2\gamma^2 n \quad (\text{provided } eB < 0) \tag{58}$$

Here we denoted by $\uparrow$ and $\downarrow$ the fermion states: $\sigma_z\,|\uparrow\rangle = 1,\ \sigma_z\,|\downarrow\rangle = -1$.

This equality of the energy values implies degeneracy of energy levels. However, there is no such degeneracy for the $|0\rangle\,|\downarrow\rangle$ state because there is no corresponding state $|-1\rangle\,|\uparrow\rangle$. All levels except the ground level are therefore double-degenerate. This has a great impact on the Quantum Hall effect in graphene.

4 Quantum Hall effect

4.1 Quantum Hall effect

4.1.1 Basics of Quantum Hall effect

Quantum Hall effect is the Quantum-mechanical version of the classical Hall effect observed in two-dimensional electron systems. Namely, at low temperatures and strong magnetic fields the Hall conductance takes on quantized values: $\sigma = \nu \frac{e^2}{h}$.

An Illustration of the Hall effect is given in Fig. 8. Circular currents induced by magnetic field add up to edge currents flowing in different directions. Voltage applied across the probe induces difference of these edge currents which leads to a total non-zero current. In practice we first produce this current and then apply voltage across the probe in order to reach equilibrium.

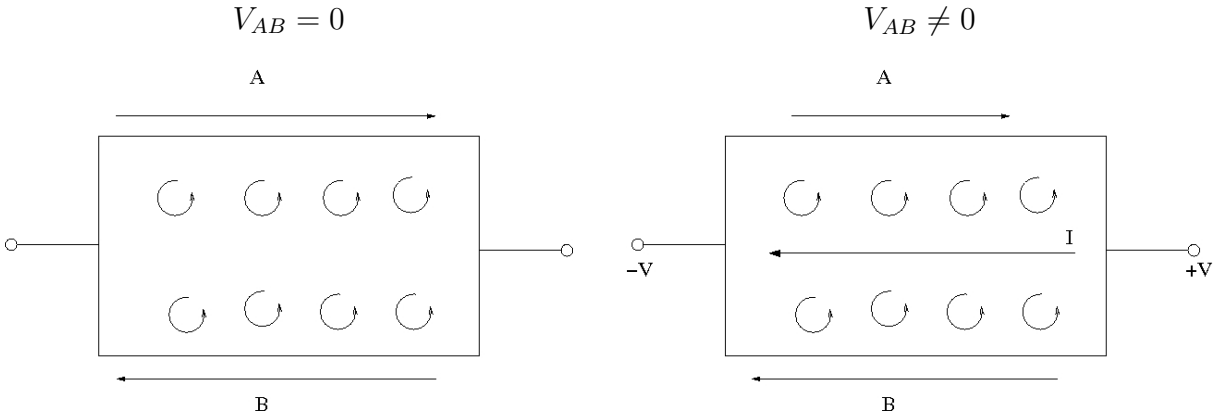

Figure 8: Edge currents in two-dimensional systems leading to Hall effect

We use the standard formulae to describe the Quantum Hall effect in our system(graphene):

$$\sigma_{xy} = -\frac{ec\rho}{B}, \quad \rho = n_e - n_h \tag{59}$$

$$\text{filling factor: } \nu_B = \frac{|\rho|}{N_B}, \quad \nu_B = \frac{\pi c\hbar}{|eB|}\,|\rho| \tag{60}$$

$$\text{Recall: } N_B = \frac{1}{2\pi l_H^2} \times 2 \tag{61}$$

From that we get the following expression for the conductance:

$$\Rightarrow \sigma_{xy} = -\frac{e^2 c}{|eB|}|\rho| sgn(eB) sgn(\mu) = -\frac{e^2}{\pi\hbar}\nu_B sgn(eB) sgn(\mu) \tag{62}$$

4.1.2 Unusual Quantum Hall effect in graphene

Our main goal is to investigate the specific properties of the Quantum Hall effect in graphene. First, let us calculate the conductance using the Bose-Einstein-distribution:

$$n_e = \frac{1}{e^{\frac{\varepsilon-\mu}{T}} + 1}, \quad n_h = 1 - n_e \tag{63}$$

$$\Rightarrow \; n_e - n_h = \tanh\frac{\mu - \varepsilon}{2T} \tag{64}$$

We have to sum up contributions from all energy levels and take care of the right degeneracy discussed above:

$$\nu_B sgn(\mu) = \frac{1}{2}\left(\tanh\frac{\mu+\Delta}{2T} + \tanh\frac{\mu-\Delta}{2T}\right. \tag{65}$$

$$\left. + 2\sum_{n=1}^{\infty}\left(\tanh\frac{\mu+\varepsilon_n}{2T} + \tanh\frac{\mu-\varepsilon_n}{2T}\right)\right) \tag{66}$$

$$\tag{67}$$

Now we recall the expression for the energy levels and the limit behaviour of tanh:

$$\varepsilon_n = \sqrt{\Delta^2 + \gamma^2 v^2 n} \tag{68}$$

$$\tanh\left(\frac{\omega}{2T}\right) \to sgn\omega \text{ for } T \to 0 \tag{69}$$

So we count conducting electrons with the right degeneracy. This is shown in Fig. 9 & 10.

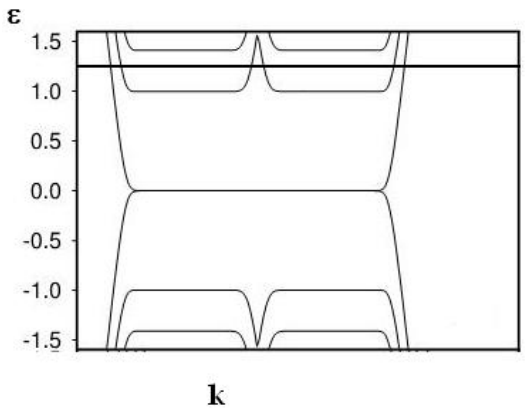

Figure 9: Intersections of the chemical potential with Landau-levels deformed near the edges correspond to the conducting electrons. Be aware of degeneracies!

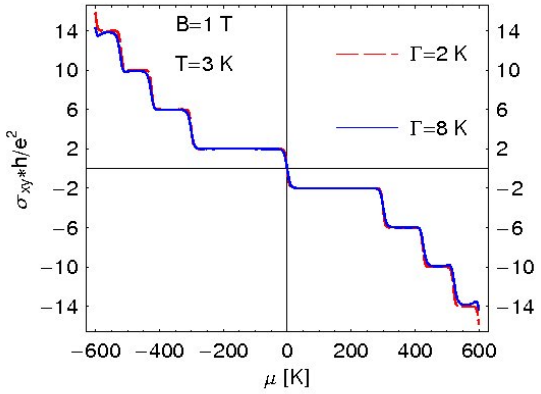

Figure 10: Unusual first half-step of σ_{xy} confirmed by the number of intersections in Fig. 9.

In spite of typical integer steps we observe a half-step in the middle and integer steps from their on. This interesting property of graphene emerges as shown above from the form of its Hamiltonian.

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