5 Asymptotic Notation

We are usually not interested in exact running times, but only in an asymtotic classification of the running time, that ignores constant factors and constant additive offsets.

- We are usually interested in the running times for large values of n. Then constant additive terms do not play an important role.
- An exact analysis (e.g. *exactly* counting the number of operations in a RAM) may be hard, but wouldn't lead to more precise results as the computational model is already quite a distance from reality.
- A linear speed-up (i.e., by a constant factor) is always possible by e.g. implementing the algorithm on a faster machine.
- Running time should be expressed by simple functions.

Asymptotic Notation

Formal Definition

Let f denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
  (set of functions that asymptotically grow not faster than f)
- $\Omega(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 : [g(n) \geq c \cdot f(n)]\}$
  (set of functions that asymptotically grow not slower than f)
- $\Theta(f) = \Omega(f) \cap \mathcal{O}(f)$
  (functions that asymptotically have the same growth as f)
- $o(f) = \{g \mid \forall c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
  (set of functions that asymptotically grow slower than f)
- $\omega(f) = \{g \mid \forall c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 : [g(n) \geq c \cdot f(n)]\}$
  (set of functions that asymptotically grow faster than f)

Asymptotic Notation

There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $g \in \mathcal{O}(f)$: $\quad 0 \leq \lim_{n\to\infty} \frac{g(n)}{f(n)} < \infty$
- $g \in \Omega(f)$: $\quad 0 < \lim_{n\to\infty} \frac{g(n)}{f(n)} \leq \infty$
- $g \in \Theta(f)$: $\quad 0 < \lim_{n\to\infty} \frac{g(n)}{f(n)} < \infty$
- $g \in o(f)$: $\quad \lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$
- $g \in \omega(f)$: $\quad \lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$

- Note that for the version of the Landau notation defined here, we assume that f and g are positive functions.
- There also exist versions for arbitrary functions, and for the case that the limes is not infinity.

Asymptotic Notation

Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).
2. People write $f(n) = \mathcal{O}(g(n))$, when they mean $f \in \mathcal{O}(g)$, with $f : \mathbb{N} \to \mathbb{R}^+, n \mapsto f(n)$, and $g : \mathbb{N} \to \mathbb{R}^+, n \mapsto g(n)$.
3. People write e.g. $h(n) = f(n) + o(g(n))$ when they mean that there exists a function $z : \mathbb{N} \to \mathbb{R}^+, n \mapsto z(n), z \in o(g)$ such that $h(n) \le f(n) + z(n)$.

2. In this context $f(n)$ does **not** mean the function f evaluated at n, but instead it is a shorthand for the function itself (leaving out domain and codomain and only giving the rule of correspondence of the function).

3. This is particularly useful if you do not want to ignore constant factors. For example the median of n elements can be determined using $\frac{3}{2}n + o(n)$ comparisons.

Asymptotic Notation

Abuse of notation

4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

Asymptotic Notation

Lemma 3

Let f, g be functions with the property
$\exists n_0 > 0\ \forall n \geq n_0 : f(n) > 0$ *(the same for g). Then*

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant c*
- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$
- $\mathcal{O}(f(n)) \cdot \mathcal{O}(g(n)) = \mathcal{O}(f(n) \cdot g(n))$
- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(\max\{f(n), g(n)\})$

The expressions also hold for Ω. Note that this means that
$f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

Asymptotic Notation

Comments

- Do not use asymptotic notation within induction proofs.
- For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.