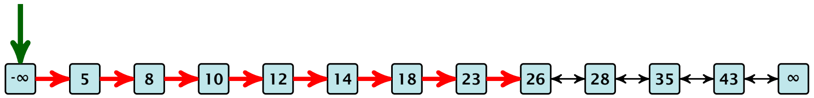
7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

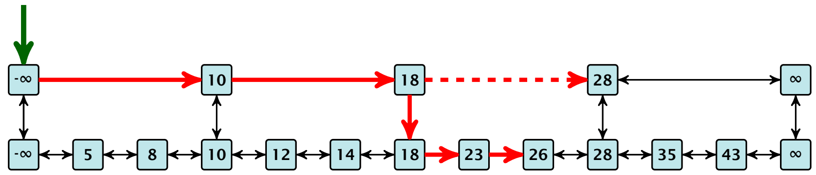
- time for search $\Theta(n)$
- time for insert $\Theta(n)$ (dominated by searching the item)
- time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(1)$



7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:



Let $|L_1|$ denote the number of elements in the "express lane", and $|L_0| = n$ the number of all elements (ignoring dummy elements).

Worst case search time: $|L_1| + \frac{|L_0|}{|L_1|}$ (ignoring additive constants)

Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

7.5 Skip Lists

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list L_{i-1}.

Search(x) ($k + 1$ lists $L_0, \ldots, L_k$)

- Find the largest item in list L_k that is smaller than x. At most $|L_k| + 2$ steps.
- Find the largest item in list L_{k-1} that is smaller than x. At most $\lceil \frac{|L_{k-1}|}{|L_k|+1} \rceil + 2$ steps.
- Find the largest item in list L_{k-2} that is smaller than x. At most $\lceil \frac{|L_{k-2}|}{|L_{k-1}|+1} \rceil + 2$ steps.
- ...
- At most $|L_k| + \sum_{i=1}^{k} \frac{L_{i-1}}{L_i} + 3(k+1)$ steps.

7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

Worst case running time is: $\mathcal{O}(r^{-k}n + kr)$. Choose

$$r = \sqrt[k+1]{n} \quad \Longrightarrow \quad \text{time: } \mathcal{O}(k \sqrt[k+1]{n})$$

Choosing $k = \Theta(\log k)$ gives a logarithmic running time.

7.5 Skip Lists

How to do insert and delete?

- If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

Use randomization instead!

7.5 Skip Lists

Insert:

- A search operation gives you the insert position for element x in every list.
- Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
- Insert x into lists $L_0, \dots, L_{t-1}$.

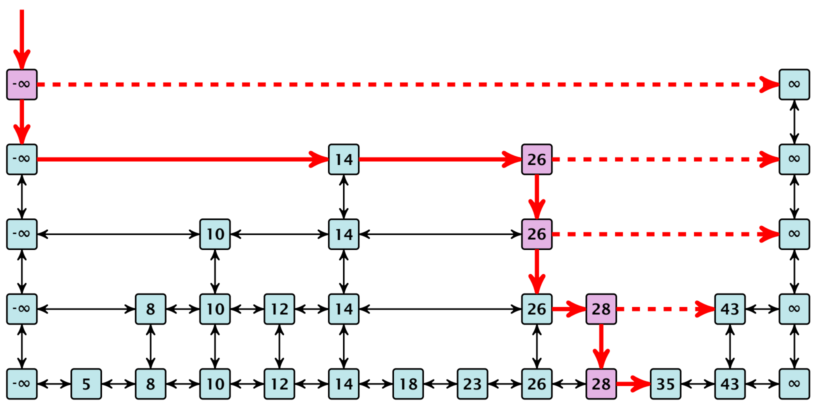
Delete:

- You get all predecessors via backward pointers.
- Delete x in all lists in actually appears in.

The time for both operation is dominated by the search time.

Skip Lists

Insert (35):



7.5 Skip Lists

Lemma 20

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

This means for any constant α the search takes time $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

Note that the constant in the $\mathcal{O}$-notation may depend on α.

High Probability

Suppose there are a polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

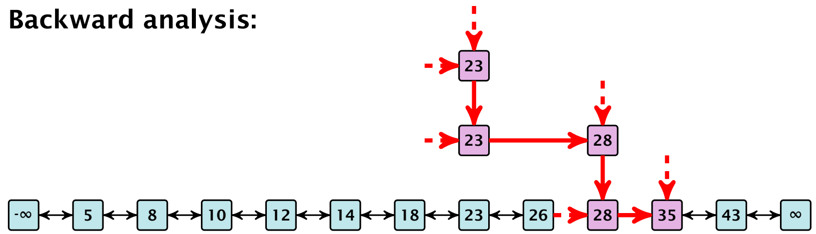
Then the probabilityx that all E_i hold is at least

$$\begin{aligned}\Pr[E_1 \wedge \cdots \wedge E_\ell] &= 1 - \Pr[\bar{E}_1 \vee \cdots \vee \bar{E}_\ell] \\ &\le 1 - n^c \cdot n^{-\alpha} \\ &= 1 - n^{c-\alpha} \;.\end{aligned}$$

This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

Skip Lists

Backward analysis:



At each point the path goes up with probability $1/2$ and left with probability $1/2$.

We show that w.h.p:

- A "long" search path must also go very high.
- There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

7.5 Skip Lists

Let $E_{z,k}$ denote the event that a search path is of length z (number of edges) but does not visit a list above L_k.

In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

7.5 Skip Lists

$$\Pr[E_{z,k}] \le \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\le \binom{z}{k} 2^{-(z-k)} \le \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \le \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \ge 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\le \left(\frac{2ez}{k}\right)^k (2^{-\beta})^k \cdot n^{-\alpha} \le \left(\frac{2e(\beta+\alpha)}{2^\beta}\right)^k n^{-\alpha}$$

now choosing $\beta = 6\alpha$ gives

$$\le \left(\frac{42\alpha}{64^\alpha}\right)^k n^{-\alpha} \le n^{-\alpha}$$

for $\alpha \ge 1$.

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the even A_{k+1} must hold.
Hence,

$$\begin{aligned}\Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)}\end{aligned}$$

This means, the search requires at most z steps, w. h. p.