# 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order $k$ with constant coefficients ($c_0, c_k \neq 0$).

- $T(n)$ only depends on the $k$ preceding values. This means the recurrence relation is of order $k$.
- The recurrence is linear as there are no products of $T[n]$'s.
- If $f(n) = 0$ then the recurrence relation becomes a linear, homogenous recurrence relation of order $k$.

# 6.3 The Characteristic Polynomial

**Observations:**

- The solution $T[0], T[1], T[2], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[0], \ldots, T[k-1]$.
- In fact, any $k$ consecutive values completely determine the solution.
- $k$ non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

**Approach:**

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

# The Homogenous Case

The solution space

$$S = \Big\{ T = T[0], T[1], T[2], \ldots \;\Big|\; T \text{ fulfills recurrence relation} \Big\}$$

is a vector space. This means that if $T_1, T_2 \in S$, then also $\alpha T_1 + \beta T_2 \in S$, for arbitrary constants $\alpha, \beta$.

**How do we find a non-trivial solution?**

We guess that the solution is of the form $\lambda^n$, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

# The Homogenous Case

Dividing by $\lambda^{n-k}$ gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

This means that if $\lambda_i$ is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the $k$ (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values $\alpha_i$.

# The Homogenous Case

## Lemma 5

*Assume that the characteristic polynomial has $k$ distinct roots $\lambda_1, \dots, \lambda_k$. Then all solutions to the recurrence relation are of the form*

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n \ .$$

## Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

# The Homogenous Case

## Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$\begin{array}{ccccccccc} \alpha_1 \cdot \lambda_1 & + & \alpha_2 \cdot \lambda_2 & + & \cdots & + & \alpha_k \cdot \lambda_k & = & T[1] \\ \alpha_1 \cdot \lambda_1^2 & + & \alpha_2 \cdot \lambda_2^2 & + & \cdots & + & \alpha_k \cdot \lambda_k^2 & = & T[2] \\ & & & & \vdots & & & & \\ \alpha_1 \cdot \lambda_1^k & + & \alpha_2 \cdot \lambda_2^k & + & \cdots & + & \alpha_k \cdot \lambda_k^k & = & T[k] \end{array}$$

# The Homogenous Case

## Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i'$s such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & \vdots & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

We show that the column vectors are linearly independent. Then the above equation has a solution.

# The Homogenous Case

## Proof (cont.).

This we show by induction:

- **base case** $(k = 1)$:
  A vector $(\lambda_i)$, $\lambda_i \neq 0$ is linearly independent.
- **induction step** $(k \to k+1)$:
  assume for contradiction that there exist $\alpha_i$'s with

$$\alpha_1 \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix} + \cdots + \alpha_k \begin{pmatrix} \lambda_k \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix} = 0$$

and not all $\alpha_i = 0$. **Then all $\boldsymbol{\alpha_i \neq 0}$!** □

# The Homogeneous Case

$$\alpha_1 \begin{pmatrix} \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix} + \cdots + \alpha_k \begin{pmatrix} \lambda_1 \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix} = 0$$

where $v_1 :=$ the first $k$ entries of the first vector, $\lambda_1 v_1 =$ its last $k$ entries; $v_k :=$ the first $k$ entries of the last vector, $\lambda_k v_k =$ its last $k$ entries.

This means that

$$\sum_{i=1}^{k} \alpha_i v_i = 0 \text{ and } \sum_{i=1}^{k} \lambda_i \alpha_i v_i = 0$$

Hence,

$$\sum_{i=1}^{k-1} \alpha_i v_i + \alpha_k v_k = 0 \text{ and } -\frac{1}{\lambda_k} \sum_{i=1}^{k-1} \lambda_i \alpha_i v_i = \alpha_k v_k$$

# The Homogeneous Case

This gives that

$$\sum_{i=1}^{k-1}(1-\frac{\lambda_i}{\lambda_k})\alpha_i v_i = 0 \enspace .$$

This is a contradiction as the $v_i$'s are linearly independent because of induction hypothesis.

# The Homogeneous Case

**What happens if the roots are not all distinct?**

Suppose we have a root $\lambda_i$ with multiplicity (Vielfachheit) at least 2. Then not only is $\lambda_i^n$ a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P(\lambda)\lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since $\lambda_i$ is a root we can write this as $Q(\lambda)(\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root $\lambda_i$.

This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$\underbrace{c_0 n\lambda_i^n}_{T[n]} + \underbrace{c_1(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + \underbrace{c_k(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

# The Homogeneous Case

Suppose $\lambda_i$ has multiplicity $j$. We know that

$$c_0 n\lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with $\lambda$; plugging in $\lambda_i$)

Doing this again gives

$$c_0 n^2\lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\ldots,j-1$.

# The Homogeneous Case

### Lemma 6

*Let $P[\lambda]$ denote the characteristic polynomial to the recurrence*

$$c_0T[n] + c_1T[n-1] + \cdots + c_kT[n-k] = 0$$

*Let $\lambda_i$, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities $\ell_i$. Then the general solution to the recurrence is given by*

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of $\alpha_{ij}$'s is a solution to the recurrence.

# Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

# Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)^n+\beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$T[0]=0$ gives $\alpha+\beta=0$.

$T[1]=1$ gives

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1\Longrightarrow\alpha-\beta=\frac{2}{\sqrt{5}}$$

# Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

# The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

# The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where $T_h$ is any solution to the homogeneous equation, and $T_p$ is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

# The Inhomogeneous Case

Example:

$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

# The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

# The Inhomogeneous Case

If $f(n)$ is a polynomial of degree $r$ this method can be applied $r+1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n-1] &= 2T[n-2] - T[n-3] + 2(n-1) - 1 \\ &= 2T[n-2] - T[n-3] + 2n - 3 \end{aligned}$$

Difference:

$$\begin{aligned} T[n] - T[n-1] =& 2T[n-1] - T[n-2] + 2n - 1 \\ & - 2T[n-2] + T[n-3] - 2n + 3 \end{aligned}$$

$$T[n] = 3T[n-1] - 3T[n-2] + T[n-3] + 2$$

and so on...