

8 Seidels LP-algorithm

- ▶ Suppose we want to solve $\min\{c^t x \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have m constraints.
- ▶ In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
- ▶ If d is much smaller than m one can do a lot better.
- ▶ In the following we develop an algorithm with running time $\mathcal{O}(d! \cdot m)$, i.e., **linear in m** .

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Setting:

- ▶ We assume an LP of the form

$$\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

- ▶ Further we assume that the LP is **non-degenerate**.
- ▶ We assume that the optimum solution is **unique**.
- ▶ We assume that the LP is **bounded**.

Ensuring Conditions

Given a **standard minimization LP**

$$\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

how can we obtain an LP of the required form?

- ▶ **Compute a lower bound on $c^t x$ for any basic feasible solution.**

Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b .

Multiply entries in A, b by s to obtain integral entries. **This does not change the feasible region.**

Add slack variables; denote the resulting matrix with \bar{A} .

If B is an optimal basis then x_B with $\bar{A}_B x_B = b$, gives an optimal assignment to the basis variables (non-basic variables are 0).

Theorem 2 (Cramers Rule)

Let M be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by

$$x_j = \frac{\det(M_j)}{\det(M)},$$

where M_j is the matrix obtained from M by replacing the j -th column by the vector b .

Proof:

- ▶ Define

$$X_j = \begin{pmatrix} | & & | & | & | \\ e_1 & \cdots & e_{j-1} & x & e_{j+1} & \cdots & e_n \\ | & & | & | & | \end{pmatrix}$$

Note that expanding along the j -th column gives that $\det(X_j) = x_j$.

- ▶ Further, we have

$$MX_j = \begin{pmatrix} | & & | & | & | \\ Me_1 & \cdots & Me_{j-1} & Mx & Me_{j+1} & \cdots & Men \\ | & & | & | & | \end{pmatrix} = M_j$$

- ▶ Hence,

$$x_j = \det(X_j) = \frac{\det(M_j)}{\det(M)}$$

Bounding the Determinant

Let Z be the maximum absolute entry occurring in A , b or c . Let C denote the matrix obtained from \bar{A}_B by replacing the j -th column with vector b .

Observe that

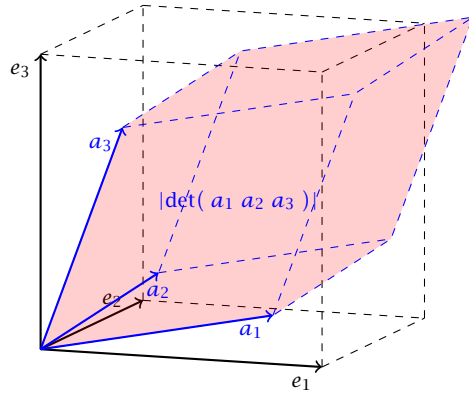
$$\begin{aligned} |\det(C)| &= \left| \sum_{\pi \in S_m} \prod_{1 \leq i \leq m} \operatorname{sgn}(\pi) C_{i\pi(i)} \right| \\ &\leq \sum_{\pi \in S_m} \prod_{1 \leq i \leq m} |C_{i\pi(i)}| \\ &\leq m! \cdot Z^m. \end{aligned}$$

Bounding the Determinant

Alternatively, Hadamard's inequality gives

$$\begin{aligned} |\det(C)| &\leq \prod_{i=1}^m \|C_{*i}\| \leq \prod_{i=1}^m (\sqrt{m}Z) \\ &\leq m^{m/2} Z^m. \end{aligned}$$

Hadamards Inequality



Hadamards inequality says that the red volume is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

Ensuring Conditions

Given a **standard minimization LP**

$$\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^t x$ for any basic feasible solution.** Add the constraint $c^t x \geq -mZ(m! \cdot Z^m) - 1$. **Note that this constraint is superfluous unless the LP is unbounded.**

Ensuring Conditions

Make the LP **non-degenerate** by perturbing the right-hand side vector b .

Make the LP solution **unique** by perturbing the optimization direction c .

Compute an optimum basis for the new LP.

- If the cost is $c^t x = -(mZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
- Otw. we have an optimum basis.

In the following we use \mathcal{H} to denote the set of all constraints apart from the constraint $c^t x \geq -mZ(m! \cdot Z^m) - 1$.

We give a routine $\text{SeidelLP}(\mathcal{H}, d)$ that is given a set \mathcal{H} of **explicit, non-degenerate** constraints over d variables, and minimizes $c^t x$ over all feasible points.

In addition it obeys the implicit constraint $c^t x \geq -(mZ)(m! \cdot Z^m) - 1$.

Algorithm 1 SeidelLP(\mathcal{H}, d)

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1: if  $d = 1$  then solve 1-dimensional problem and return;
2: if  $\mathcal{H} = \emptyset$  then return  $x$  on implicit constraint hyperplane
3: choose random constraint  $h \in \mathcal{H}$ 
4:  $\hat{\mathcal{H}} \leftarrow \mathcal{H} \setminus \{h\}$ 
5:  $\hat{x}^* \leftarrow \text{SeidelLP}(\hat{\mathcal{H}}, d)$ 
6: if  $\hat{x}^* = \text{infeasible}$  then return infeasible
7: if  $\hat{x}^*$  fulfills  $h$  then return  $\hat{x}^*$ 
8: // optimal solution fulfills  $h$  with equality, i.e.,  $A_h x = b_h$ 
9: solve  $A_h x = b_h$  for some variable  $x_\ell$ ;
10: eliminate  $x_\ell$  in constraints from  $\hat{\mathcal{H}}$  and in implicit constr.;
11:  $\hat{x}^* \leftarrow \text{SeidelLP}(\hat{\mathcal{H}}, d - 1)$ 
12: if  $\hat{x}^* = \text{infeasible}$  then
13:     return infeasible
14: else
15:     add the value of  $x_\ell$  to  $\hat{x}^*$  and return the solution
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- ▶ If $d = 1$ we can solve the 1-dimensional problem in time $\mathcal{O}(m)$.
- ▶ If $d > 1$ and $m = 0$ we take time $\mathcal{O}(d)$ to return d -dimensional vector x .
- ▶ The first recursive call takes time $T(m - 1, d)$ for the call plus $\mathcal{O}(d)$ for checking whether the solution fulfills h .
- ▶ If we are unlucky and \hat{x}^* does not fulfill h we need time $\mathcal{O}(d(m + 1)) = \mathcal{O}(dm)$ to eliminate x_ℓ . Then we make a recursive call that takes time $T(m - 1, d - 1)$.
- ▶ The probability of being unlucky is at most d/m as there are at most d constraints whose removal will decrease the objective function (recall that the solution is unique).

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This gives the recurrence

$$T(m, d) = \begin{cases} \mathcal{O}(m) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m - 1, d) + \\ \frac{d}{m} (\mathcal{O}(dm) + T(m - 1, d - 1)) & \text{otw.} \end{cases}$$

Note that $T(m, d)$ denotes the **expected running time**.

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Let C be the largest constant in the \mathcal{O} -notations.

We show $T(m, d) \leq C f(d) \max\{1, m\}$.

$d = 1$:

$$T(m, 1) \leq Cm \leq C f(1) \max\{1, m\} \text{ for } f(1) \geq 1$$

$d > 1; m = 0$:

$$T(0, d) \leq \mathcal{O}(d) \leq Cd \leq C f(d) \max\{1, m\} \text{ for } f(d) \geq d$$

$d > 1; m = 1$:

$$\begin{aligned} T(1, d) &= \mathcal{O}(d) + T(0, d) + d(\mathcal{O}(d) + T(0, d - 1)) \\ &\leq Cd + Cd + Cd^2 + dT(0, d - 1) \\ &\leq C f(d) \max\{1, m\} \text{ for } f(d) \geq 4d^2 \end{aligned}$$

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$d > 1; m > 1 :$

(by induction hypothesis statm. true for $d' < d, m' \geq 0;$

and for $d' = d, m' < m$)

$$\begin{aligned} T(m, d) &= \mathcal{O}(d) + T(m-1, d) + \frac{d}{m} (\mathcal{O}(dm) + T(m-1, d-1)) \\ &\leq Cd + Cf(d)(m-1) + Cd^2 + \frac{d}{m} Cf(d-1)(m-1) \\ &\leq 2Cd^2 + Cf(d)(m-1) + dCf(d-1) \\ &\leq Cf(d)m \end{aligned}$$

if $f(d) \geq df(d-1) + 2d^2$.

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► Define $f(1) = 4 \cdot 1^2$ and $f(d) = df(d-1) + 4d^2$ for $d > 1$.

Then

$$\begin{aligned} f(d) &= 4d^2 + df(d-1) \\ &= 4d^2 + d[4(d-1)^2 + (d-1)f(d-2)] \\ &= 4d^2 + d[4(d-1)^2 + (d-1)[4(d-2)^2 + (d-2)f(d-3)]] \\ &= 4d^2 + 4d(d-1)^2 + 4d(d-1)(d-2)^2 + \dots \\ &\quad + 4d(d-1)(d-2) \cdot \dots \cdot 4 \cdot 3 \cdot 1^2 \\ &= 4d! \left(\frac{d^2}{d!} + \frac{(d-1)^2}{(d-1)!} + \frac{(d-2)^2}{(d-2)!} + \dots \right) \\ &= \mathcal{O}(d!) \end{aligned}$$

since $\sum_{i \geq 1} \frac{i^2}{i!}$ is a constant.