**Lemma 9**
$\mathrm{OPT}(I') \le \mathrm{OPT}(I) \le \mathrm{OPT}(I') + k$

**Proof 2:**

- Any bin packing for $I'$ gives a bin packing for $I$ as follows.
- Pack the items of group $1$ into $k$ new bins;
- Pack the items of groups $2$, where in the packing for $I'$ the items for group $2$ have been packed;
- ...



Assume that our instance does not contain pieces smaller than $\epsilon/2$. Then $\mathrm{SIZE}(I) \ge \epsilon n/2$.

We set $k = \lfloor \epsilon \mathrm{SIZE}(I) \rfloor$.

Then $n/k \le 2n/\lfloor \epsilon^2 n/2 \rfloor \le 4/\epsilon^2$ (here we used $\lfloor \alpha \rfloor \ge \alpha/2$ for $\alpha \ge 1$).

Hence, after grouping we have a constant number of piece sizes ($4/\epsilon^2$) and at most a constant number ($2/\epsilon$) can fit into any bin.

We can find an optimal packing for such instances by the previous Dynamic Programming approach.

- cost (for large items) at most

$$\mathrm{OPT}(I') + k \le \mathrm{OPT}(I) + \epsilon \mathrm{SIZE}(I) \le (1+\epsilon)\mathrm{OPT}(I)$$

- running time $\mathcal{O}((\frac{2}{\epsilon} n)^{4/\epsilon^2})$.

**Can we do better?**

In the following we show how to obtain a solution where the number of bins is only

$$\mathrm{OPT}(I) + \mathcal{O}(\log^2(\mathrm{SIZE}(I))) \ .$$

Note that this is usually better than a guarantee of

$$(1+\epsilon)\mathrm{OPT}(I) + 1 \ .$$



## Configuration LP

**Change of Notation:**

- Group pieces of identical size.
- Let $s_1$ denote the largest size, and let $b_1$ denote the number of pieces of size $s_1$.
- $s_2$ is second largest size and $b_2$ number of pieces of size $s_2$;
- ...
- $s_m$ smallest size and $b_m$ number of pieces of size $s_m$.

## Configuration LP

A possible packing of a bin can be described by an $m$-tuple $(t_1, \ldots, t_m)$, where $t_i$ describes the number of pieces of size $s_i$. Clearly,

$$\sum_i t_i \cdot s_i \le 1 \ .$$

We call a vector that fulfills the above constraint a configuration.



## Configuration LP

Let $N$ be the number of configurations (exponential).

Let $T_1, \ldots, T_N$ be the sequence of all possible configurations (a configuration $T_j$ has $T_{ji}$ pieces of size $s_i$).

$$\begin{array}{llrcl}
\min & & \sum_{j=1}^{N} x_j & & \\
\text{s.t.} & \forall i \in \{1 \ldots m\} & \sum_{j=1}^{N} T_{ji} x_j & \ge & b_i \\
& \forall j \in \{1, \ldots, N\} & x_j & \ge & 0 \\
& \forall j \in \{1, \ldots, N\} & x_j & \text{integral} &
\end{array}$$



**How to solve this LP?**

later...



We can assume that each item has size at least $1/\text{SIZE}(I)$.

## Harmonic Grouping

- Sort items according to size (monotonically decreasing).
- Process items in this order; close the current group if size of items in the group is at least $2$ (or larger). Then open new group.
- I.e., $G_1$ is the smallest cardinality set of largest items s.t. total size sums up to at least $2$. Similarly, for $G_2, \ldots, G_{r-1}$.
- Only the size of items in the last group $G_r$ may sum up to less than $2$.



## Harmonic Grouping

From the grouping we obtain instance $I'$ as follows:

- Round all items in a group to the size of the largest group member.
- Delete all items from group $G_1$ and $G_r$.
- For groups $G_2, \ldots, G_{r-1}$ delete $n_i - n_{i-1}$ items.
- Observe that $n_i \geq n_{i-1}$.



**Lemma 10**
*The number of different sizes in $I'$ is at most $\mathrm{SIZE}(I)/2$.*

- Each group that survives (recall that $G_1$ and $G_r$ are deleted) has total size at least $2$.
- Hence, the number of surviving groups is at most $\mathrm{SIZE}(I)/2$.
- All items in a group have the same size in $I'$.



**Lemma 11**
*The total size of deleted items is at most $\mathcal{O}(\log(\mathrm{SIZE}(I)))$.*

- The total size of items in $G_1$ and $G_r$ is at most $6$ as a group has total size at most $3$.
- Consider a group $G_i$ that has strictly more items than $G_{i-1}$.
- It discards $n_i - n_{i-1}$ pieces of total size at most
$$3\frac{n_i - n_{i-1}}{n_i} \leq \sum_{j=n_{i-1}+1}^{n_i} \frac{3}{j}$$
since the smallest piece has size at most $3/n_i$.
- Summing over all $i$ that have $n_i > n_{i-1}$ gives a bound of at most
$$\sum_{j=1}^{n_{r-1}} \frac{3}{j} \leq \mathcal{O}(\log(\mathrm{SIZE}(I))) \enspace .$$
(note that $n_r \leq \mathrm{SIZE}(I)$ since we assume that the size of each item is at least $1/\mathrm{SIZE}(I)$).

**Algorithm 1** BinPack

**if** $\text{SIZE}(I) < 10$ **then**
    pack remaining items greedily
Apply harmonic grouping to create instance $I'$; pack discarded items in at most $\mathcal{O}(\log(\text{SIZE}(I)))$ bins.
Let $x$ be optimal solution to configuration LP
Pack $\lfloor x_j \rfloor$ bins in configuration $T_j$ for all $j$; call the packed instance $I_1$.
Let $I_2$ be remaining pieces from $I'$
Pack $I_2$ via BinPack$(I_2)$



## Analysis

$$\text{OPT}_{\text{LP}}(I_1) + \text{OPT}_{\text{LP}}(I_2) \le \text{OPT}_{\text{LP}}(I') \le \text{OPT}_{\text{LP}}(I)$$

**Proof:**

- Each piece surviving in $I'$ can be mapped to a piece in $I$ of no lesser size. Hence, $\text{OPT}_{\text{LP}}(I') \le \text{OPT}_{\text{LP}}(I)$
- $\lfloor x_j \rfloor$ is feasible solution for $I_1$ (even integral).
- $x_j - \lfloor x_j \rfloor$ is feasible solution for $I_2$.



## Analysis

Each level of the recursion partitions pieces into three types

1. Pieces discarded at this level.
2. Pieces scheduled because they are in $I_1$.
3. Pieces in $I_2$ are handed down to the next level.

Pieces of type 2 summed over all recursion levels are packed into at most $\text{OPT}_{\text{LP}}$ many bins.

Pieces of type 1 are packed into at most

$$\mathcal{O}(\log(\text{SIZE}(I))) \cdot L$$

many bins where $L$ is the number of recursion levels.



## Analysis

We can show that $\text{SIZE}(I_2) \le \text{SIZE}(I)/2$. Hence, the number of recursion levels is only $\mathcal{O}(\log(\text{SIZE}(I_{\text{original}})))$ in total.

- The number of non-zero entries in the solution to the configuration LP for $I'$ is at most the number of constraints, which is the number of different sizes ($\le \text{SIZE}(I)/2$).
- The total size of items in $I_2$ can be at most $\sum_{j=1}^{N} x_j - \lfloor x_j \rfloor$ which is at most the number of non-zero entries in the solution to the configuration LP.

## How to solve the LP?

Let $T_1, \dots, T_N$ be the sequence of all possible configurations (a configuration $T_j$ has $T_{ji}$ pieces of size $s_i$).
In total we have $b_i$ pieces of size $s_i$.

**Primal**

$$\begin{array}{llrcl} \min & & \sum_{j=1}^{N} x_j & & \\ \text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\ & \forall j \in \{1, \dots, N\} & x_j & \geq & 0 \end{array}$$

**Dual**

$$\begin{array}{llrcl} \max & & \sum_{i=1}^{m} y_i b_i & & \\ \text{s.t.} & \forall j \in \{1, \dots, N\} & \sum_{i=1}^{m} T_{ji} y_i & \leq & 1 \\ & \forall i \in \{1, \dots, m\} & y_i & \geq & 0 \end{array}$$



## Separation Oracle

Suppose that I am given variable assignment $y$ for the dual.

**How do I find a violated constraint?**

I have to find a configuration $T_j = (T_{j1}, \dots, T_{jm})$ that

- is feasible, i.e.,
$$\sum_{i=1}^{m} T_{ji} \cdot s_i \leq 1 \ ,$$
- and has a large profit
$$\sum_{i=1}^{m} T_{ji} y_i > 1$$

But this is the Knapsack problem.



## Separation Oracle

We have FPTAS for Knapsack. This means if a constraint is violated with $1 + \epsilon' = 1 + \frac{\epsilon}{1-\epsilon}$ we find it, since we can obtain at least $(1 - \epsilon)$ of the optimal profit.

The solution we get is feasible for:

**Dual′**

$$\begin{array}{llrcl} \max & & \sum_{i=1}^{m} y_i b_i & & \\ \text{s.t.} & \forall j \in \{1, \dots, N\} & \sum_{i=1}^{m} T_{ji} y_i & \leq & 1 + \epsilon' \\ & \forall i \in \{1, \dots, m\} & y_i & \geq & 0 \end{array}$$

**Primal′**

$$\begin{array}{llrcl} \min & & (1 + \epsilon') \sum_{j=1}^{N} x_j & & \\ \text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\ & \forall j \in \{1, \dots, N\} & x_j & \geq & 0 \end{array}$$

## Separation Oracle

If the value of the computed dual solution (which may be infeasible) is $z$ then

$$\mathrm{OPT} \leq z \leq (1 + \epsilon')\mathrm{OPT}$$

**How do we get good primal solution (not just the value)?**

- The constraints used when computing $z$ certify that the solution is feasible for $\mathrm{DUAL}'$.
- Suppose that we drop all unused constraints in DUAL. We will compute the same solution feasible for $\mathrm{DUAL}'$.
- Let $\mathrm{DUAL}''$ be DUAL without unused constraints.
- The dual to $\mathrm{DUAL}''$ is PRIMAL where we ignore variables for which the corresponding dual constraint has not been used.
- The optimum value for $\mathrm{PRIMAL}''$ is at most $(1 + \epsilon')\mathrm{OPT}$.
- We can compute the corresponding solution in polytime.

This gives that overall we need at most

$$(1+\epsilon')\mathrm{OPT}_{\mathrm{LP}}(I) + \mathcal{O}(\log^2(\mathrm{SIZE}(I)))$$

bins.

We can choose $\epsilon' = \frac{1}{\mathrm{OPT}}$ as $\mathrm{OPT} \leq$ #items and since we have a fully polynomial time approximation scheme (FPTAS) for knapsack.