# 6.4 Generating Functions

### Definition 4 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} a_n z^n;$$

- exponential generating function (exponentielle Erzeugendenfunktion) is

$$F(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n.$$


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**Example 5**

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There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** $f$ and $g$ are equal if $a_n = b_n$ for all $n$.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c = \sum_{p=0}^{n} a_p b_{n-p}$.

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The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

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What does $\sum_{n\geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n\geq 0}^{\infty} z^n\right) = 1 \ .$$

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

# 6.4 Generating Functions

Computing the $k$-th derivative of $\sum z^n$.

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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$$\sum_{n\geq 0} nz^n = \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n$$

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# 6.4 Generating Functions

$$\begin{aligned}\sum_{n\geq 0} nz^n &= \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n \\ &= \frac{1}{(1-z)^2} - \frac{1}{1-z} \\ &= \frac{z}{(1-z)^2}\end{aligned}$$

The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

# 6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.


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## Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

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A(z) &= \sum_{n\geq 0} a_n z^n \\
&= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \\
&= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n
\end{aligned}
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&= z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} z^n \\
&= zA(z) + \sum_{n\geq 0} z^n \\
&= zA(z) + \frac{1}{1-z}
\end{aligned}$$


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Solving for $A(z)$ gives

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Hence, $a_n = n + 1$.

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
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| | |
| | |
| | |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $1$ | $\frac{1}{1-z}$ |
| | |
| | |
| | |
| | |
| | |
| | |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $1$ | $\frac{1}{1-z}$ |
| $n+1$ | $\frac{1}{(1-z)^2}$ |
| | |
| | |
| | |
| | |
| | |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $1$ | $\frac{1}{1-z}$ |
| $n+1$ | $\frac{1}{(1-z)^2}$ |
| $\binom{n+k}{k}$ | $\frac{1}{(1-z)^{k+1}}$ |
| | |
| | |
| | |
| | |

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| $n+1$ | $\frac{1}{(1-z)^2}$ |
| $\binom{n+k}{k}$ | $\frac{1}{(1-z)^{k+1}}$ |
| $n$ | $\frac{z}{(1-z)^2}$ |
| | |
| | |
| | |

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| $1$ | $\frac{1}{1-z}$ |
| $n+1$ | $\frac{1}{(1-z)^2}$ |
| $\binom{n+k}{k}$ | $\frac{1}{(1-z)^{k+1}}$ |
| $n$ | $\frac{z}{(1-z)^2}$ |
| $a^n$ | $\frac{1}{1-az}$ |
| | |
| | |

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| $a^n$ | $\frac{1}{1-az}$ |
| $n^2$ | $\frac{z(1+z)}{(1-z)^3}$ |
| | |

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| $a^n$ | $\frac{1}{1-az}$ |
| $n^2$ | $\frac{z(1+z)}{(1-z)^3}$ |
| $\frac{1}{n!}$ | $e^z$ |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
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# Some Generating Functions

| *n*-th sequence element | generating function |
|---|---|
| $cf_n$ | $cF$ |
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| | |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $cf_n$ | $cF$ |
| $f_n + g_n$ | $F + G$ |
| | |
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| | |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $cf_n$ | $cF$ |
| $f_n + g_n$ | $F + G$ |
| $\sum_{i=0}^{n} f_i g_{n-i}$ | $F \cdot G$ |
| | |
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| | |
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| $f_{n-k}\quad (n \ge k);\quad 0$ otw. | $z^k F$ |
| $\sum_{i=0}^{n} f_i$ | $\dfrac{F(z)}{1-z}$ |
| | |
| | |


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| $\sum_{i=0}^{n} f_i$ | $\dfrac{F(z)}{1-z}$ |
| $nf_n$ | $z\dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$ |
| | |

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| $\sum_{i=0}^{n} f_i$ | $\dfrac{F(z)}{1-z}$ |
| $nf_n$ | $z\dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$ |
| $c^n f_n$ | $F(cz)$ |

# Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.

# Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

# Solving Recursions with Generating Functions

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6. The coefficients of the resulting power series are the $a_n$.

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$$\begin{aligned} A(z) &= 1 + \sum_{n\geq 1}(2a_{n-1})z^n \\ &= 1 + 2z\sum_{n\geq 1} a_{n-1}z^{n-1} \\ &= 1 + 2z\sum_{n\geq 0} a_n z^n \end{aligned}$$

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$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} a_n z^n \\ &= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n \end{aligned}$$

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A(z) &= \sum_{n\ge 0} a_n z^n \\
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&= 1 + 3z \sum_{n\ge 1} a_{n-1} z^{n-1} + \sum_{n\ge 1} n z^n
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&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
\end{aligned}
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$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

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$$z^2 - z + 1 = A(1-z)^2 + B(1-3z)(1-z) + C(1-3z)$$

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This gives

$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \end{aligned}$$

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This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

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This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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**5.** Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2}$$

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$$= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1) z^n$$

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&= \frac{7}{4} \cdot \sum_{n\geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\geq 0} z^n - \frac{1}{2} \cdot \sum_{n\geq 0} (n+1) z^n \\
&= \sum_{n\geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \right) z^n
\end{aligned}
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&= \sum_{n\ge0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big)z^n \\
&= \sum_{n\ge0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{2}n - \frac{3}{4}\Big)z^n
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&= \sum_{n\geq 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big)z^n \\
&= \sum_{n\geq 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{2}n - \frac{3}{4}\Big)z^n
\end{aligned}
$$

6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.