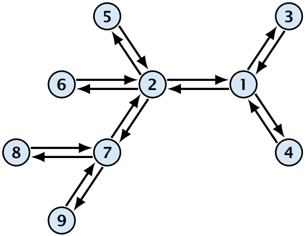
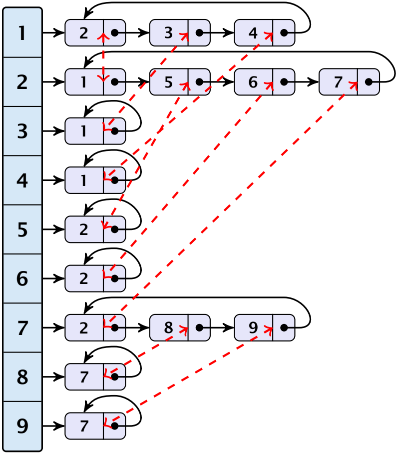
Tree Algorithms





Euler Circuits

Every node v fixes an arbitrary ordering among its adjacent nodes:

$$u_0, u_1, \ldots, u_{d-1}$$

We obtain an Euler tour by setting

$$\mathrm{succ}((u_i, v)) = (v, u_{(i+1) \bmod d})$$

Euler Circuits

Lemma 1

An Euler circuit can be computed in constant time $\mathcal{O}(1)$ *with* $\mathcal{O}(n)$ *operations.*

Euler Circuits — Applications

Rooting a tree

- split the Euler tour at node r
- this gives a list on the set of directed edges (Euler path)
- assign $x[e] = 1$ for every edge;
- perform parallel prefix; let $s[\cdot]$ be the result array
- if $s[(u,v)] < s[(v,u)]$ then u is parent of v;

Euler Circuits — Applications

Postorder Numbering

- split the Euler tour at node r
- this gives a list on the set of directed edges (Euler path)
- assign $x[e] = 1$ for every edge $(v, \text{parent}(v))$
- assign $x[e] = 0$ for every edge $(\text{parent}(v), v)$
- perform parallel prefix
- $\text{post}(v) = s[(v, \text{parent}(v))]$; $\text{post}(r) = n$

Euler Circuits — Applications

Level of nodes

- split the Euler tour at node r
- this gives a list on the set of directed edges (Euler path)
- assign $x[e] = -1$ for every edge $(v, \text{parent}(v))$
- assign $x[e] = 1$ for every edge $(\text{parent}(v), v)$
- perform parallel prefix
- $\text{level}(v) = s[(\text{parent}(v), v)]$; $\text{level}(r) = 0$

Euler Circuits — Applications

Number of descendants

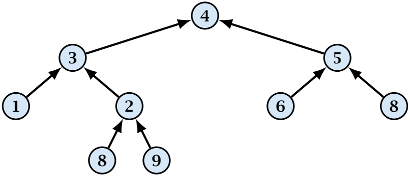
- split the Euler tour at node r
- this gives a list on the set of directed edges (Euler path)
- assign $x[e] = 0$ for every edge $(\text{parent}(v), v)$
- assign $x[e] = 1$ for every edge $(v, \text{parent}(v))$, $v \neq r$
- perform parallel prefix
- $\text{size}(v) = s[(v, \text{parent}(v))] - s[(\text{parent}(v), v)]$

Rake Operation

Given a binary tree T.

Given a leaf $u \in T$ with $p(u) \neq r$ the rake-operation does the following

- remove u and $p(u)$
- attach sibling of u to $p(p(u))$



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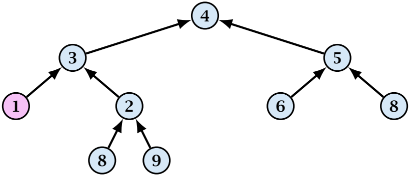

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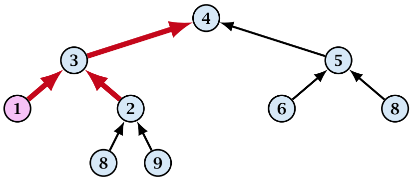

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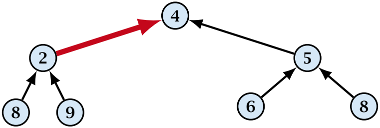

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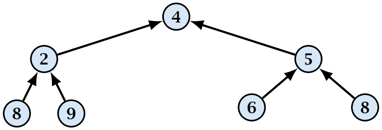

We want to apply rake operations to a binary tree T until T just consists of the root with two children.

Possible Problems:

- we could concurrently apply the rake-operation to two siblings
- we could concurrently apply the rake-operation to two leaves u and v such that $p(u)$ and $p(v)$ are connected

By choosing leaves carefully we ensure that none of the above cases occurs

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Algorithm:

- label leaves consecutively from left to right (excluding left-most and right-most leaf), and store them in an array A
- for $\lceil \log(n+1) \rceil$ iterations
  - apply rake to all odd leaves that are left children
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- the rake operation does not change the order of leaves
- two leaves that are siblings do not perform a rake operation in the same round because one is even and one odd at the start of the round
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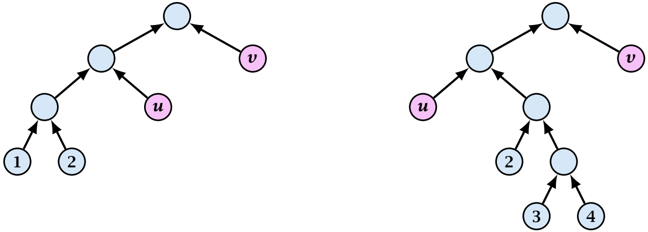
Cases, when the left edge btw. $p(u)$ and $p(v)$ is a left-child edge.



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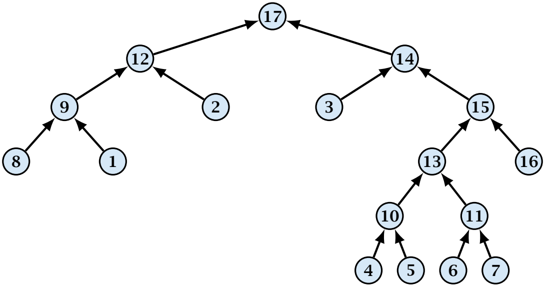

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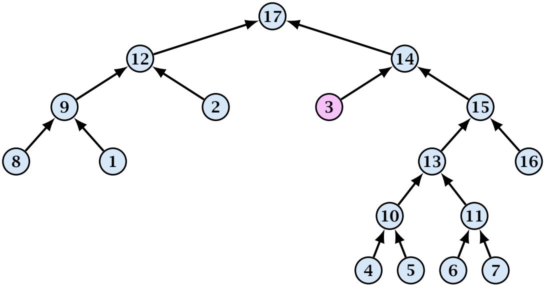

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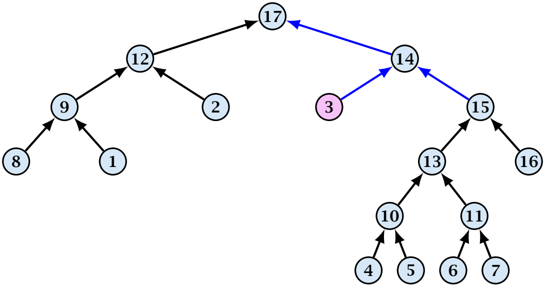

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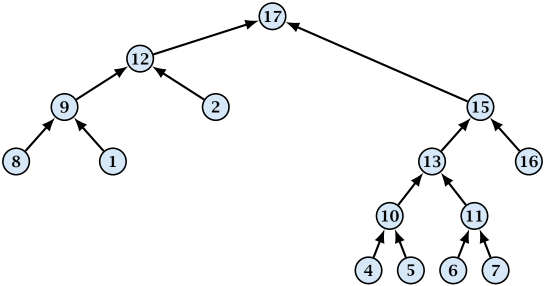

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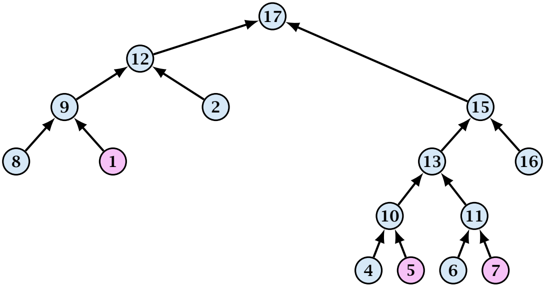

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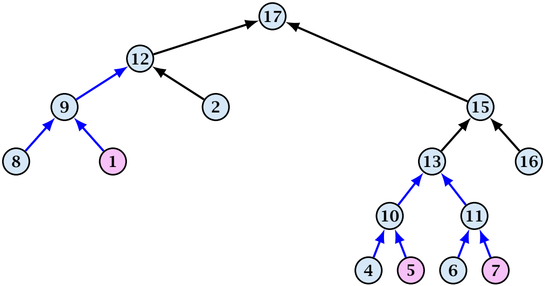

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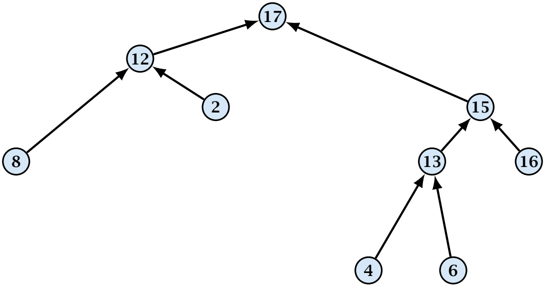

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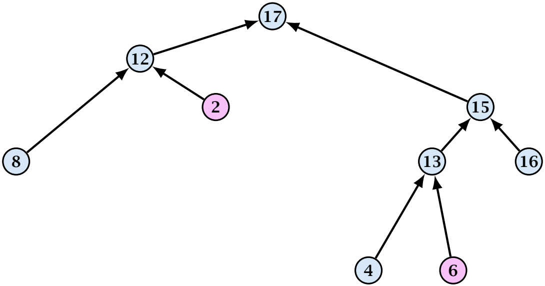

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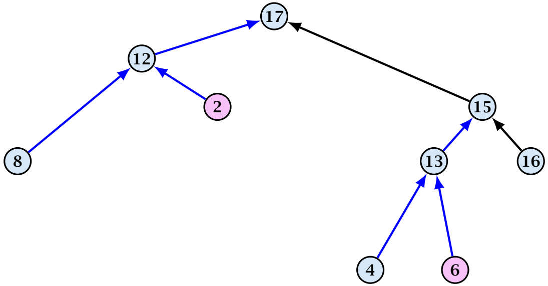

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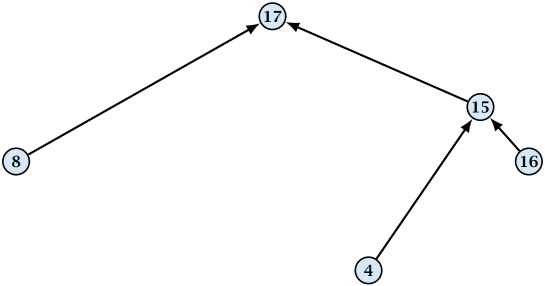

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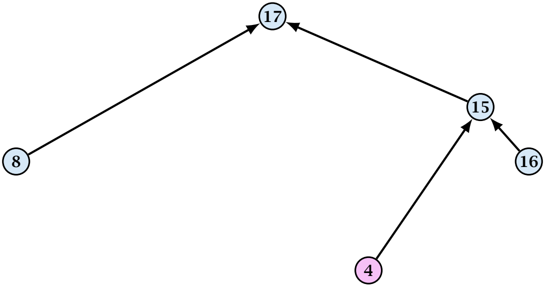

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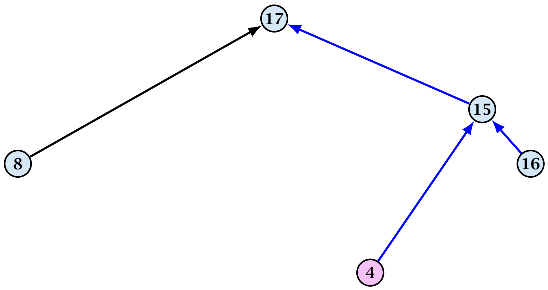

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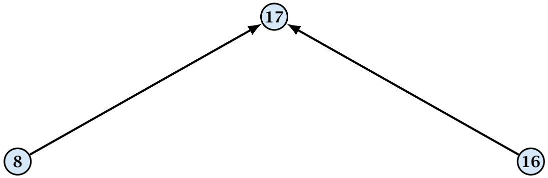

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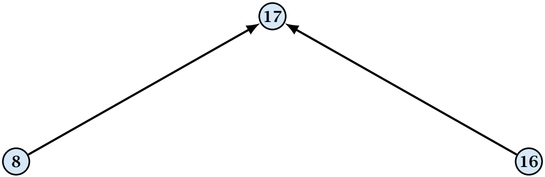

- one iteration can be performed in constant time with $\mathcal{O}(|A|)$ processors, where A is the array of leaves;
- hence, **all** iterations can be performed in $\mathcal{O}(\log n)$ time and $\mathcal{O}(n)$ work;
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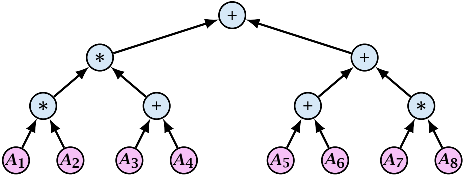

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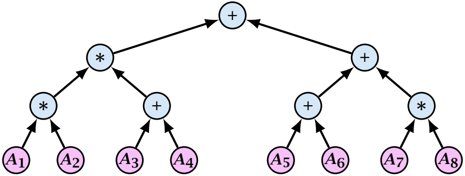

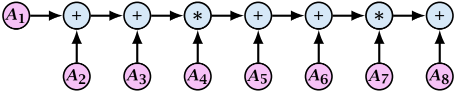

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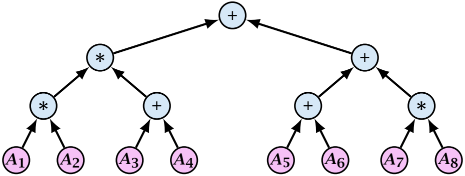

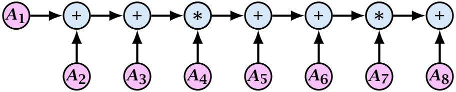

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We can use the rake-operation to do this quickly.

Applying the rake-operation changes the tree.

In order to maintain the value we introduce parameters a_v and b_v for every node that still allows to compute the value of a node based on the value of its children.

Invariant:
Let u be internal node with children v and w. Then

$$\mathrm{val}(u) = (a_v \cdot \mathrm{val}(v) + b_v) \otimes (a_w \cdot \mathrm{val}(w) + b_w)$$

where $\otimes \in \{*, +\}$ is the operation at node u.

Initially, we can choose $a_v = 1$ and $b_v = 0$ for every node.

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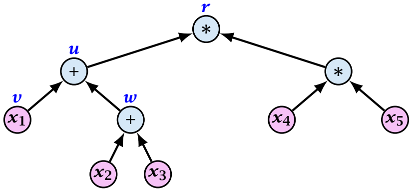

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$$\begin{aligned}\mathrm{val}(u) &= (a_v \cdot \mathrm{val}(v) + b_v) + (a_w \cdot \mathrm{val}(w) + b_w) \\ &= x_1 + (a_w \cdot \mathrm{val}(w) + b_w)\end{aligned}$$

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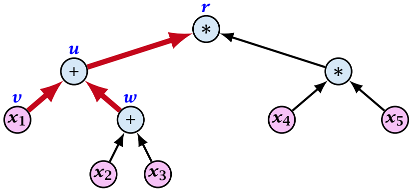

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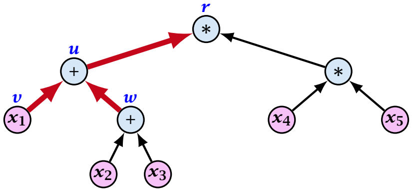

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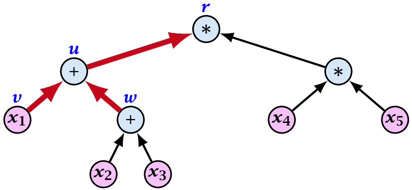

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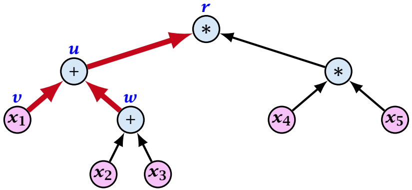

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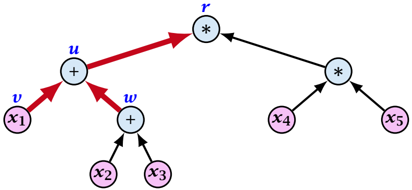

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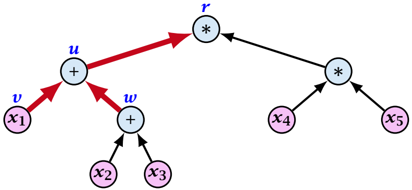

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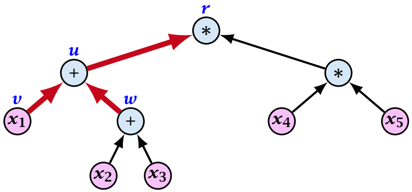

Currently the value at u is

$$\begin{aligned}\mathrm{val}(u) &= (a_v \cdot \mathrm{val}(v) + b_v) + (a_w \cdot \mathrm{val}(w) + b_w) \\ &= x_1 + (a_w \cdot \mathrm{val}(w) + b_w)\end{aligned}$$

In the expression for r this goes in as

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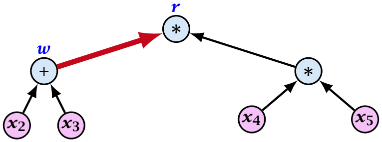

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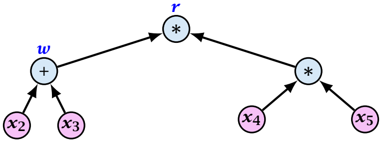

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If we change the a and b-values during a rake-operation according to the previous slide we can calculate the value of the root in the end.

Lemma 2

We can evaluate an arithmetic expression tree in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(n)$ regardless of the height or depth of the tree.

By performing the rake-operation in the reverse order we can also compute the value at each node in the tree.

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Lemma 3

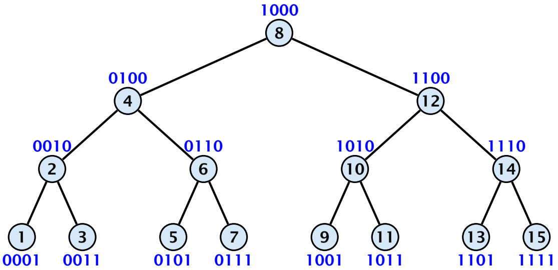
We compute tree functions for arbitrary trees in time $\mathcal{O}(\log n)$ and a linear number of operations.

proof on board...

In the LCA (least common ancestor) problem we are given a tree and the goal is to design a data-structure that answers LCA-queries in constant time.

Least Common Ancestor

LCAs on complete binary trees (inorder numbering):

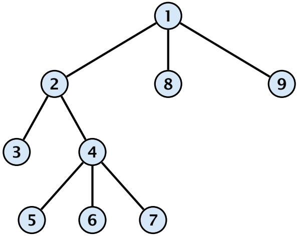


The least common ancestor of u and v is

$$z_1 z_2 \ldots z_i 1 0 \ldots 0$$

where z_{i+1} is the first bit-position in which u and v differ.

Least Common Ancestor



nodes	1	2	3	2	4	5	4	6	4	7	4	2	1	8	1	9	1
levels	0	1	2	1	2	3	2	3	2	3	2	1	0	1	0	1	0

$\ell(v)$ is index of first appearance of v in node-sequence.

$r(v)$ is index of last appearance of v in node-squence.

$\ell(v)$ and $r(v)$ can be computed in constant time, given the node- and level-sequence.

Least Common Ancestor

Lemma 4

1. *u is ancestor of v iff $\ell(u) < \ell(v) < r(u)$*
2. *u and v are not related iff either $r(u) < \ell(v)$ or $\ell(u) < r(v)$*
3. *suppose $r(u) < \ell(v)$ then $\mathrm{LCA}(u, v)$ is vertex with minimum level over interval $[r(u), \ell(v)]$.*

Range Minima Problem

Given an array $A[1 \dots n]$, a range minimum query (ℓ, r) consists of a left index $\ell \in \{1, \dots, n\}$ and a right index $r \in \{1, \dots, n\}$.

The answer has to return the index of the minimum element in the subsequence $A[\ell \dots r]$.

The goal in the range minima problem is to preprocess the array such that range minima queries can be answered quickly (constant time).

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Observation

Given an algorithm for solving the range minima problem in time $T(n)$ and work $W(n)$ we can obtain an algorithm that solves the LCA-problem in time $\mathcal{O}(T(n) + \log n)$ and work $\mathcal{O}(n + W(n))$.

Remark

In the sequential setting the LCA-problem and the range minima problem are equivalent. This is not necessarily true in the parallel setting.

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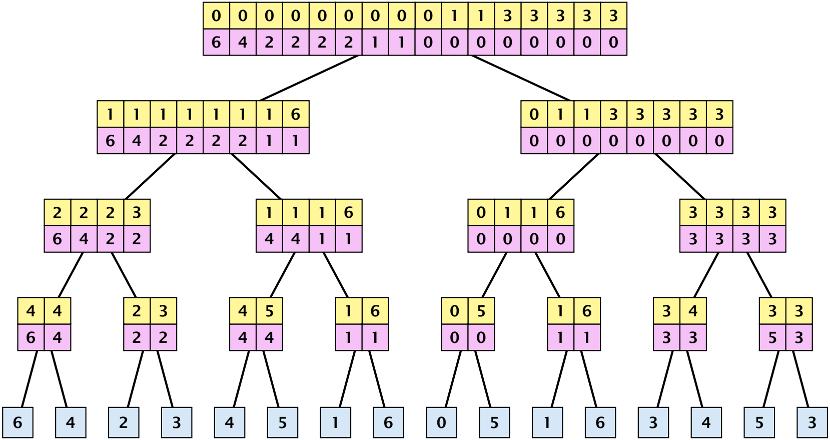
Prefix and Suffix Minima

Tree with prefix-minima and suffix-minima:



- Suppose we have an array A of length $n = 2^k$
- We compute a complete binary tree T with n leaves.
- Each internal node corresponds to a subsequence of A. It contains an array with the prefix and suffix minima of this subsequence.

Given the tree T we can answer a range minimum query (ℓ, r) in constant time.

- we can determine the LCA x of ℓ and r in constant time since T is a complete binary tree.
- Then we consider the suffix minimum of ℓ in the left child of x and the prefix minimum of r in the right child of x.
- The minimum of these two values is the result.

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Lemma 5

We can solve the range minima problem in time $\mathcal{O}(\log n)$ *and work* $\mathcal{O}(n \log n)$.

Reducing the Work

Partition A into blocks B_i of length $\log n$

Preprocess each B_i block separately by a sequential algorithm so that range-minima queries within the block can be answered in constant time. (**how?**)

For each block B_i compute the minimum x_i and its prefix and suffix minima.

Use the previous algorithm on the array $(x_1, \ldots, x_{n/\log n})$.

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Answering a query (ℓ, r):

- if ℓ and r are from the same block the data-structure for this block gives us the result in constant time
- if ℓ and r are from different blocks the result is a minimum of three elements:
  - the suffix minmum of entry ℓ in ℓ's block
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