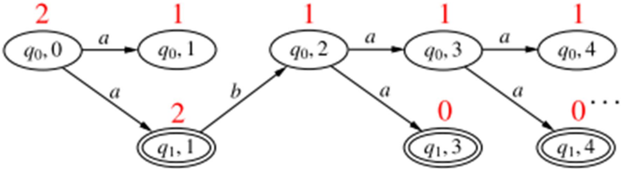
Solving the first problem

- We use owing states and breakpoints again:
  - A breakpoint of a ranking is now a level of the ranking such that no state of the level owes a visit to a node of odd rank.
  - We have again: a ranking is odd iff it has infinitely many breakpoints.
  - We enrich the state with a set of owing states, and choose the accepting states as those in which the set is empty.

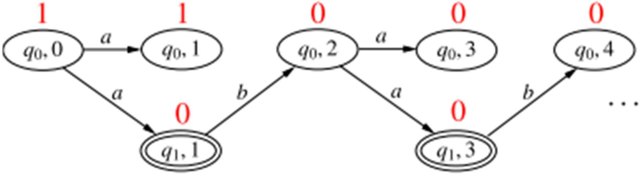
Owing states



$$\begin{bmatrix}2\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\ 2\end{bmatrix} \xrightarrow{b} \begin{bmatrix}1\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\ 0\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\ 0\end{bmatrix} \cdots$$

$\{q_0\}$ $\{q_1\}$ $\emptyset$ $\{q_1\}$ $\emptyset$

Owing rankings



$$\begin{bmatrix}1\\\bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\0\end{bmatrix} \xrightarrow{b} \begin{bmatrix}0\\\bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}0\\0\end{bmatrix} \xrightarrow{b} \begin{bmatrix}0\\\bot\end{bmatrix} \cdots$$

$\emptyset$ $\{q_1\}$ $\{q_0\}$ $\{q_0, q_1\}$ $\{q_0\}$

Second draft for $\bar{A}$

- For a two-state A (the case of more states is analogous):
  - States: all pairs $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, O$ wher accepting states get even rank, and O is set of owing states (of even rank)
  - Initial states: all $\begin{bmatrix} n_1 \\ \bot \end{bmatrix}, \{q_0\}$ where n_1 even if q_0 accepting.
  - Transitions: all $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, O \xrightarrow{a} \begin{bmatrix} n_1' \\ n_2' \end{bmatrix}, O'$ s.t. ranks don‘t increase and owing states are correctly updated
  - Final states: all states $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \emptyset$

- The runs of $\bar{A}$ on a word w correspond to all the rankings of $dag(w)$.
- The accepting runs of $\bar{A}$ on a word w correspond to all the odd rankings of $dag(w)$.
- Therefore: $L(\bar{A}) = \overline{L(A)}$

Solving the second problem

Proposition: If w is rejected by A, then $dag(w)$ has an odd ranking in which ranks are taken from the range $[0,2n]$, where n is the number of states of A. Further, the initial node gets rank $2n$.

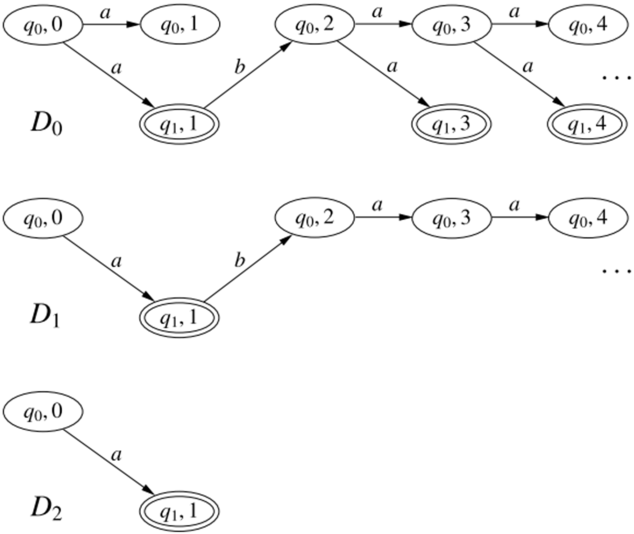
Proof: We construct such a ranking as follows:

- we proceed in $n+1$ rounds (from round 0 to round n), each round with two steps $k.0$ and $k.1$ with the exception of round n which only has $n.0$
- each step removes a set of nodes together with all its descendants.
- the nodes removed at step $i.j$ get rank $2i+j$
- the rank of the initial node is increased to $2n$ if necessary (preserves the properties of rankings).

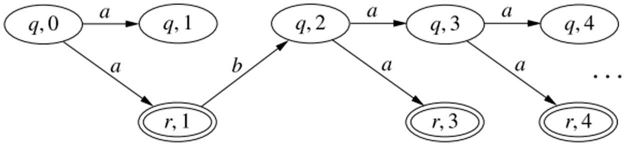
The steps

- Step $i.0$: remove all nodes having only finitely many successors.
- Step $i.1$: remove nodes that are non-accepting and have no accepting descendants

- This immediately guarantees :
  1. Ranks along a path cannot increase.
  2. Accepting states get even ranks, because they can only be removed at step $i.0$
- It remains to prove: no nodes left after $n + 1$ rounds .


q0, 0
a
q0, 1
q0, 2
q0, 3
q0, 4
b
q1, 1
q1, 3
q1, 4
D0
D1
D2
...

- To prove: no nodes left after n rounds .
- Each level of a dag has a width



- We define the width of a dag as the largest level width that appears infinitely often.
- Each round decreases the width of the dag by at least 1.
- Since the intial width is at most n after at most n rounds the width is 0, and then step $n.0$ removes all nodes.

Final $\bar{A}$

- For a two-state A (the case of more states is analogous):
  - States: all pairs $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, O$ where O set of owing states and accepting states get even rank
  - Initial state: all $\begin{bmatrix} 2n \\ \bot \end{bmatrix}, \{q_0\}$
  - Transitions: all $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, O \xrightarrow{a} \begin{bmatrix} n_1' \\ n_2' \end{bmatrix}, O'$ s.t. ranks don't increase and owing states are correctly updated
  - Final states: all states $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \emptyset$

An example

- We construct the complements of

  $A_1 = (\{q\}, \{a\}, \delta, \{q\}, \{q\})$ with $\delta(q, a) = \{q\}$

  $A_2 = (\{q\}, \{a\}, \delta, \{q\}, \emptyset)$ with $\delta(q, a) = \{q\}$

- States of A_1:

  $\langle 0, \emptyset \rangle, \langle 2, \emptyset \rangle, \langle 0, \{q\} \rangle, \langle 2, \{q\} \rangle$

- States of A_2:

  $\langle 0, \emptyset \rangle, \langle 1, \emptyset \rangle, \langle 2, \emptyset \rangle, \langle 0, \{q\} \rangle, \langle 2, \{q\} \rangle$

- Initial state of A_1 and A_2: $\langle 2, \{q\} \rangle$

An example

- Transitions of A_1:
  $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 2, \{q\}\rangle$, $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 0, \emptyset\rangle$, $\langle 0, \{q\}\rangle \xrightarrow{a} \langle 0, \{q\}\rangle$
- Transitions of A_2:
  $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 2, \{q\}\rangle$, $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 1, \emptyset\rangle$, $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 0, \emptyset\rangle$,
  $\langle 1, \emptyset\rangle \xrightarrow{a} \langle 1, \emptyset\rangle$, $\langle 1, \emptyset\rangle \xrightarrow{a} \langle 0, \{q\}\rangle$,
  $\langle 0, \{q\}\rangle \xrightarrow{a} \langle 0, \{q\}\rangle$
- Final states of A_1: $\langle 0, \emptyset\rangle, \langle 2, \emptyset\rangle$ (unreachable)
- Final states of A_2: $\langle 0, \emptyset\rangle, \langle 1, \emptyset\rangle, \langle 2, \emptyset\rangle$ (only $\langle 1, \emptyset\rangle$ is reachable)

CompNBA(A)

Input: NBA $A = (Q, \Sigma, \delta, q_0, F)$

Output: NBA $\overline{A} = (\overline{Q}, \Sigma, \overline{\delta}, \overline{q}_0, \overline{F})$ with $L_\omega(\overline{A}) = \overline{L_\omega(A)}$

$\overline{Q}, \overline{\delta}, \overline{F} \leftarrow \emptyset$
$\overline{q}_0 \leftarrow [lr_0, \{q_0\}]$
$W \leftarrow \{\ [lr_0, \{q_0\}]\ \}$
while $W \neq \emptyset$ **do**
    pick $[lr, P]$ **from** W; **add** $[lr, P]$ **to** $\overline{Q}$
    if $P = \emptyset$ **then add** $[lr, P]$ **to** $\overline{F}$
    for all $a \in \Sigma, lr' \in \mathfrak{R}$ such that $lr \overset{a}{\mapsto} lr'$ **do**
        if $P \neq \emptyset$ **then** $P' \leftarrow \{q \in \delta(P, a) \mid lr'(q) \text{ is even}\}$
        else $P' \leftarrow \{q \in Q \mid lr'(q) \text{ is even}\}$
        add $([lr, P], a, [lr', P'])$ **to** $\overline{\delta}$
        if $[lr', P'] \notin \overline{Q}$ **then add** $[lr', P']$ **to** W
return $(\overline{Q}, \Sigma, \overline{\delta}, \overline{q}_0, \overline{F})$

Complexity

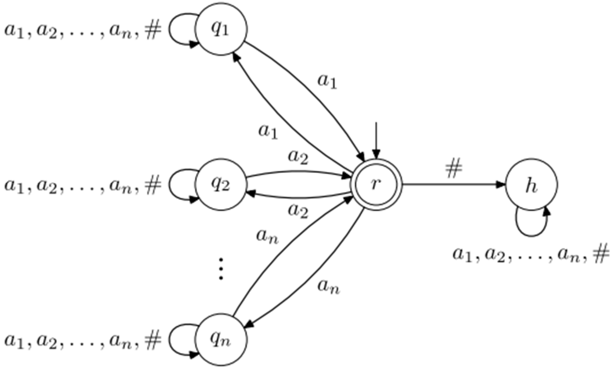
- A state consists of a level of a ranking and a set of owing states.
- A level assigns to each state a number f $[0,2n]$ or the symbol $\bot$.
- So the complement NBA has at most $(2n+2)^n \cdot 2^n \in n^{O(n)} = 2^{O(n \log n)}$ states.
- Compare with 2^n for the NFA case.
- We show that the $\log n$ factor is unavoidable.

We define a family $\{L_n\}_{n\geq 1}$ of ω-languages s.t.

- L_n is accepted by a NBA with $n+2$ states.
- Every NBA accepting $\overline{L_n}$ has at least $n! \in 2^{\Theta(n \log n)}$ states.

- The alphabet of L_n is $\Sigma_n = \{1,2,\ldots,n,\#\}$.
- Assign to a word $w \in \Sigma_n$ a graph $G(w)$ as follows:
  - Vertices: the numbers $1,2,\ldots,n$.
  - Edges: there is an edge $i \to j$ iff w contains infinitely many occurrences of ij.
- Define: $w \in L_n$ iff $G(w)$ has a cycle.

- L_n is accepted by a NBA with $n + 2$ states.



Every NBA accepting $\overline{L_n}$ has at least $n! \in 2^{\Theta(n \log n)}$ states.

- Let τ denote a permutation of $1, 2, \ldots, n$.
- We have:
  a) For every τ, the word $(\tau\,\#)^\omega$ belongs to $\overline{L_n}$ (i.e., its graph contains no cycle).
  b) For every two distinct τ_1, τ_2, every word containing inf. many occurrences of τ_1 and inf. many occurrences of τ_2 belongs to L_n.

Every NBA accepting $\overline{L_n}$ has at least $n! \in 2^{\Theta(n \log n)}$ states.

- Assume A recognizes $\overline{L_n}$ and let τ_1, τ_2 distinct. By (a), A has runs ρ_1, ρ_2 accepting $(\tau\_1 \#)^\omega$, $(\tau_2 \#)^\omega$. The sets of accepting states visited i.o. by ρ_1, ρ_2 are disjoint.
  - Otherwise we can ``interleave'' ρ_1, ρ_2 to yield an acepting run for a word with inf. Many occurrences of τ_1, τ_2, contradicting (b).
- So A has at least one accepting state for each permutation, and so at least $n!$ States.