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Change of Notation:

- Group pieces of identical size.
- Let s_1 denote the largest size, and let b_1 denote the number of pieces of size s_1.
- s_2 is second largest size and b_2 number of pieces of size s_2;
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We can assume that each item has size at least $1/\mathrm{SIZE}(I)$.

Harmonic Grouping

- Sort items according to size (monotonically decreasing).
- Process items in this order; close the current group if size of items in the group is at least 2 (or larger). Then open new group.
- I.e., G_1 is the smallest cardinality set of largest items s.t. total size sums up to at least 2. Similarly, for $G_2, \dots, G_{r-1}$.
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From the grouping we obtain instance I' as follows:

- Round all items in a group to the size of the largest group member.
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The number of different sizes in I' is at most $\mathrm{SIZE}(I)/2$.

Each group that survives (recall that G_1 and G_k are deleted) has total size at least 2.

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The total size of deleted items is at most $\mathcal{O}(\log(\mathrm{SIZE}(I)))$.

- The total size of items in G_1 and G_ℓ is at most 6 as a group has total size at most 3.
- Consider a group G_i that has strictly more items than G_{i-1}.
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$$3\frac{n_i - n_{i-1}}{n_i} \le \sum_{j=n_{i-1}+1}^{n_i} \frac{3}{j}$$

since the smallest piece has size at most $3/n_i$.

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Algorithm 1 BinPack

if $\text{SIZE}(I) < 10$ **then**
    pack remaining items greedily
Apply harmonic grouping to create instance I'; pack discarded items in at most $\mathcal{O}(\log(\text{SIZE}(I)))$ bins.
Let x be optimal solution to configuration LP
Pack $\lfloor x_j \rfloor$ bins in configuration T_j for all j; call the packed instance I_1.
Let I_2 be remaining pieces from I'
Pack I_2 via BinPack(I_2)

Analysis

$$\mathrm{OPT}_{\mathrm{LP}}(I_1) + \mathrm{OPT}_{\mathrm{LP}}(I_2) \le \mathrm{OPT}_{\mathrm{LP}}(I') \le \mathrm{OPT}_{\mathrm{LP}}(I)$$

Proof:

- Each piece surviving in I' can be mapped to a piece in I of no lesser size. Hence, $\mathrm{OPT}_{\mathrm{LP}}(I') \le \mathrm{OPT}_{\mathrm{LP}}(I)$
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Each level of the recursion partitions pieces into three types

1. Pieces discarded at this level.
2. Pieces scheduled because they are in I_1.
3. Pieces in I_2 are handed down to the next level.

Pieces of type 2 summed over all recursion levels are packed into at most $\mathrm{OPT}_{\mathrm{LP}}$ many bins.

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Let $T_1, \dots, T_N$ be the sequence of all possible configurations (a configuration T_j has T_{ji} pieces of size s_i).

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$$\begin{array}{llll} \min & & \sum_{j=1}^{N} x_j & \\ \text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq b_i \\ & \forall j \in \{1, \dots, N\} & x_j & \geq 0 \end{array}$$

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Suppose that I am given variable assignment y for the dual.

How do I find a violated constraint?

I have to find a configuration $T_j = (T_{j1}, \dots, T_{jm})$ that

- is feasible, i.e.,

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- and has a large profit

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But this is the Knapsack problem.

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The solution we get is feasible for:

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We have FPTAS for Knapsack. This means if a constraint is violated with $1 + \epsilon' = 1 + \frac{\epsilon}{1-\epsilon}$ we find it, since we can obtain at least $(1-\epsilon)$ of the optimal profit.

The solution we get is feasible for:

Dual′

$$\begin{array}{llrcl} \max & & \sum_{i=1}^{m} y_i b_i & & \\ \text{s.t.} & \forall j \in \{1,\ldots,N\} & \sum_{i=1}^{m} T_{ji} y_i & \leq & 1+\epsilon' \\ & \forall i \in \{1,\ldots,m\} & y_i & \geq & 0 \end{array}$$

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If the value of the computed dual solution (which may be infeasible) is z then

$$\mathrm{OPT} \leq z \leq (1 + \epsilon')\mathrm{OPT}$$

How do we get good primal solution (not just the value)?

- The constraints used when computing z certify that the solution is feasible for DUAL'.
- Suppose that we drop all unused constraints in DUAL. We will compute the same solution feasible for DUAL'.
- Let DUAL'' be DUAL without unused constraints.
- The dual to DUAL'' is PRIMAL where we ignore variables for which the corresponding dual constraint has not been used.
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This gives that overall we need at most

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