

Technique 5: Randomized Rounding

One round of randomized rounding:

Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

Version A: Repeat rounds until you have a cover.

Version B: Repeat for s rounds. If you have a cover STOP.
Otherwise, repeat the whole algorithm.

Technique 5: Randomized Rounding

One round of randomized rounding:

Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

Version A: Repeat rounds until you have a cover.

Version B: Repeat for s rounds. If you have a cover STOP.
Otherwise, repeat the whole algorithm.

Technique 5: Randomized Rounding

One round of randomized rounding:

Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

Version A: Repeat rounds until you have a cover.

Version B: Repeat for s rounds. If you have a cover STOP.
Otherwise, repeat the whole algorithm.

Probability that $u \in U$ is not covered (in one round):

$\Pr[u \text{ not covered in one round}]$

Probability that $u \in U$ is not covered (in one round):

$\Pr[u \text{ not covered in one round}]$

$$= \prod_{j:u \in S_j} (1 - x_j)$$

Probability that $u \in U$ is not covered (in one round):

$$\begin{aligned} & \Pr[u \text{ not covered in one round}] \\ &= \prod_{j:u \in S_j} (1 - x_j) \leq \prod_{j:u \in S_j} e^{-x_j} \end{aligned}$$

Probability that $u \in U$ is not covered (in one round):

$\Pr[u \text{ not covered in one round}]$

$$\begin{aligned} &= \prod_{j:u \in S_j} (1 - x_j) \leq \prod_{j:u \in S_j} e^{-x_j} \\ &= e^{-\sum_{j:u \in S_j} x_j} \end{aligned}$$

Probability that $u \in U$ is not covered (in one round):

$\Pr[u \text{ not covered in one round}]$

$$\begin{aligned} &= \prod_{j:u \in S_j} (1 - x_j) \leq \prod_{j:u \in S_j} e^{-x_j} \\ &= e^{-\sum_{j:u \in S_j} x_j} \leq e^{-1} . \end{aligned}$$

Probability that $u \in U$ is not covered (in one round):

$$\begin{aligned}\Pr[u \text{ not covered in one round}] &= \prod_{j:u \in S_j} (1 - x_j) \leq \prod_{j:u \in S_j} e^{-x_j} \\ &= e^{-\sum_{j:u \in S_j} x_j} \leq e^{-1} .\end{aligned}$$

Probability that $u \in U$ is not covered (after ℓ rounds):

$$\Pr[u \text{ not covered after } \ell \text{ round}] \leq \frac{1}{e^\ell} .$$

$\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}]$

$$\begin{aligned} & \Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\ &= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \dots \vee u_n \text{ not covered}] \end{aligned}$$

$$\begin{aligned} & \Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\ &= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \dots \vee u_n \text{ not covered}] \\ &\leq \sum_i \Pr[u_i \text{ not covered after } \ell \text{ rounds}] \end{aligned}$$

$$\begin{aligned} & \Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\ &= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \dots \vee u_n \text{ not covered}] \\ &\leq \sum_i \Pr[u_i \text{ not covered after } \ell \text{ rounds}] \leq ne^{-\ell} . \end{aligned}$$

$$\begin{aligned} & \Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\ &= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \dots \vee u_n \text{ not covered}] \\ &\leq \sum_i \Pr[u_i \text{ not covered after } \ell \text{ rounds}] \leq ne^{-\ell} . \end{aligned}$$

Lemma 5

With high probability $\mathcal{O}(\log n)$ rounds suffice.

$$\begin{aligned} & \Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\ &= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \dots \vee u_n \text{ not covered}] \\ &\leq \sum_i \Pr[u_i \text{ not covered after } \ell \text{ rounds}] \leq ne^{-\ell} . \end{aligned}$$

Lemma 5

With high probability $\mathcal{O}(\log n)$ rounds suffice.

With high probability:

For any constant α the number of rounds is at most $\mathcal{O}(\log n)$ with probability at least $1 - n^{-\alpha}$.

Proof: We have

$$\Pr[\text{\#rounds} \geq (\alpha + 1) \ln n] \leq ne^{-(\alpha+1)\ln n} = n^{-\alpha} .$$

Expected Cost

- ▶ Version A.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply take for each element u the cheapest set that contains u .

Expected Cost

- ▶ Version A.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply take for each element u the cheapest set that contains u .

$E[\text{cost}]$

Expected Cost

- ▶ Version A.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply take for each element u the cheapest set that contains u .

$$E[\text{cost}] \leq (\alpha + 1) \ln n \cdot \text{cost}(LP) + (n \cdot \text{OPT}) n^{-\alpha}$$

Expected Cost

- ▶ Version A.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply take for each element u the cheapest set that contains u .

$$E[\text{cost}] \leq (\alpha + 1) \ln n \cdot \text{cost}(LP) + (n \cdot \text{OPT}) n^{-\alpha} = \mathcal{O}(\ln n) \cdot \text{OPT}$$

Expected Cost

- ▶ Version B.
Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply repeat the whole process.

$$E[\text{cost}] =$$

Expected Cost

- ▶ Version B.
Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply repeat the whole process.

$$E[\text{cost}] = \Pr[\text{success}] \cdot E[\text{cost} \mid \text{success}] \\ + \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]$$

Expected Cost

- ▶ Version B.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply repeat the whole process.

$$E[\text{cost}] = \Pr[\text{success}] \cdot E[\text{cost} \mid \text{success}] \\ + \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]$$

This means

$$E[\text{cost} \mid \text{success}]$$

Expected Cost

- ▶ Version B.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply repeat the whole process.

$$E[\text{cost}] = \Pr[\text{success}] \cdot E[\text{cost} \mid \text{success}] \\ + \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]$$

This means

$$E[\text{cost} \mid \text{success}] \\ = \frac{1}{\Pr[\text{succ.}]} \left(E[\text{cost}] - \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}] \right)$$

Expected Cost

► Version B.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply repeat the whole process.

$$E[\text{cost}] = \Pr[\text{success}] \cdot E[\text{cost} \mid \text{success}] \\ + \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]$$

This means

$$E[\text{cost} \mid \text{success}] \\ = \frac{1}{\Pr[\text{succ.}]} \left(E[\text{cost}] - \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}] \right) \\ \leq \frac{1}{\Pr[\text{succ.}]} E[\text{cost}] \leq \frac{1}{1 - n^{-\alpha}} (\alpha + 1) \ln n \cdot \text{cost}(\text{LP})$$

Expected Cost

- ▶ Version B.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply repeat the whole process.

$$E[\text{cost}] = \Pr[\text{success}] \cdot E[\text{cost} \mid \text{success}] \\ + \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]$$

This means

$$E[\text{cost} \mid \text{success}] \\ = \frac{1}{\Pr[\text{succ.}]} \left(E[\text{cost}] - \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}] \right) \\ \leq \frac{1}{\Pr[\text{succ.}]} E[\text{cost}] \leq \frac{1}{1 - n^{-\alpha}} (\alpha + 1) \ln n \cdot \text{cost}(\text{LP}) \\ \leq 2(\alpha + 1) \ln n \cdot \text{OPT}$$

Expected Cost

► Version B.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply repeat the whole process.

$$E[\text{cost}] = \Pr[\text{success}] \cdot E[\text{cost} \mid \text{success}] \\ + \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]$$

This means

$$E[\text{cost} \mid \text{success}] \\ = \frac{1}{\Pr[\text{succ.}]} \left(E[\text{cost}] - \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}] \right) \\ \leq \frac{1}{\Pr[\text{succ.}]} E[\text{cost}] \leq \frac{1}{1 - n^{-\alpha}} (\alpha + 1) \ln n \cdot \text{cost}(\text{LP}) \\ \leq 2(\alpha + 1) \ln n \cdot \text{OPT}$$

for $n \geq 2$ and $\alpha \geq 1$.

Randomized rounding gives an $\mathcal{O}(\log n)$ approximation. The running time is polynomial with high probability.

Theorem 6 (without proof)

There is no approximation algorithm for set cover with approximation guarantee better than $\frac{1}{2} \log n$ unless NP has quasi-polynomial time algorithms (algorithms with running time $2^{\text{poly}(\log n)}$).

Randomized rounding gives an $\mathcal{O}(\log n)$ approximation. The running time is polynomial with high probability.

Theorem 6 (without proof)

There is no approximation algorithm for set cover with approximation guarantee better than $\frac{1}{2} \log n$ unless NP has quasi-polynomial time algorithms (algorithms with running time $2^{\text{poly}(\log n)}$).

Integrality Gap

The **integrality gap** of the SetCover LP is $\Omega(\log n)$.

- ▶ $n = 2^k - 1$
- ▶ Elements are all vectors \vec{x} over $GF[2]$ of length k (excluding zero vector).
- ▶ Every vector \vec{y} defines a set as follows

$$S_{\vec{y}} := \{\vec{x} \mid \vec{x}^T \vec{y} = 1\}$$

- ▶ each set contains 2^{k-1} vectors; each vector is contained in 2^{k-1} sets
- ▶ $x_i = \frac{1}{2^{k-1}} = \frac{2}{n+1}$ is fractional solution.

Integrality Gap

Every collection of $p < k$ sets does not cover all elements.

Hence, we get a gap of $\Omega(\log n)$.

Techniques:

- ▶ Deterministic Rounding
- ▶ Rounding of the Dual
- ▶ Primal Dual
- ▶ Greedy
- ▶ Randomized Rounding
- ▶ Local Search
- ▶ Rounding Data + Dynamic Programming