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Problem Set 02
April 20, 2014

Complexity Theory

Due date: April 27, 2014 **before** *class!*

Problem 1 (10 Points)

Define a two-dimensional Turing machine to be a TM where each of its tapes is an infinite grid (and the machine can move not only **L**eft and **R**ight but also **U**p and **D**own). Show that for every (time-constructible) $T : \mathbb{N} \to \mathbb{N}$ and every Boolean function f, if f can be computed in time $T(n)$ using a two-dimensional TM then $f \in \mathbf{DTIME}(T(n)^2)$ (on a one-dimensional TM).

Problem 2 (10 Points)

A partial function from $\{0,1\}$ to $\{0,1\}$ is a function that is not necessarily defined on all its inputs. We say that a TM M computes a partial function f if for every x on which f is defined, $M(x) = f(x)$ and for every x on which f is not defined, M gets into an infinite loop when executed on input x. If S is a set of partial functions, we define f_S to be the Boolean function that on input α outputs 1 iff M_α computes a partial function in S. Rice's Theorem says that for every nontrivial S (a set that is neither the empty set nor the set of all partial functions computable by some Turing machine), the function f_S is not computable.

1. Show that Rice's Theorem yields an alternative proof for the statement that the function HALT is not computable.
2. Prove Rice's Theorem.

Problem 3 (10 Points)

Recall that normally we assume that numbers are represented as strings using the binary basis. However, we could use other encoding schemes, for examples a representation of n in base b, denoted by $\llcorner n \lrcorner_b$ is obtained as follows: First, represent n as a sequence of digits in $\{0, \dots, b-1\}$, and then replace each digit $d \in \{0, \dots, b-1\}$ by its binary representation. You have already seen the unary representation $\llcorner n \lrcorner_1$.

1. Show that choosing a different base of representation will make no difference to the class $\mathcal{P}$. That is, show that for every subset S of the natural numbers, if we define $L_S^b = \{\llcorner n \lrcorner_b : n \in S\}$, then for every $b \geq 2, L_S^b \in \mathcal{P}$ if and only if $L_S^2 \in \mathcal{P}$.
2. Show that choosing the unary representation may make a difference by showing that the following language is in $\mathcal{P}$:

$$\textsc{UnaryFactoring} = \{(\llcorner n \lrcorner_1, \llcorner \ell \lrcorner_1, \llcorner k \lrcorner_1) : \text{ there is a prime } j \in (\ell, k) \text{ dividing } n\}.$$

Problem 4 (10 Points)

Prove or disprove that the following problems are in $\mathcal{P}$:

1. $\textsc{Halt} = \{(\alpha, x) : \text{the TM } M_\alpha \text{ halts on input } x\}$.
2. $\textsc{Tree} = \{G : G \text{ is a representation of a tree}\}$.
3. $\textsc{Halt}_\epsilon = \{\alpha : \text{the TM } M_\alpha \text{ that halts on input } \epsilon\}$.
4. Given $A \in \mathbb{R}^{n \times n}, b, y \in \mathbb{R}^n$, define
   $\text{LP} = \{(A, b, y) : y \text{ is a solution to the set of linear equations } Ax = b\}$.
5. $\textsc{Halt}_{\text{in time}} = \{(\alpha, x, t) : \text{the TM } M_\alpha \text{ halts on input } x \text{ in at most } t \text{ time steps}\}$.
6. $\textsc{Halt}_{\text{out of time}} = \{(\alpha, x, t) : \text{the TM } M_\alpha \text{ halts on input } x \text{ in at least } t \text{ time steps}\}$.