

Splay Trees

Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

Splay Trees:

- + after access, an element is moved to the root; $\text{splay}(x)$ repeated accesses are faster
- only amortized guarantee
- read-operation change the tree

Splay Trees

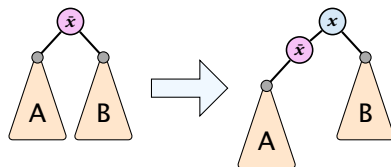
find(x)

- ▶ search for x according to a search tree
- ▶ let \bar{x} be last element on search-path
- ▶ $\text{splay}(\bar{x})$

Splay Trees

insert(x)

- ▶ search for x ; \bar{x} is last visited element during search (successor or predecessor of x)
- ▶ $\text{splay}(\bar{x})$ moves \bar{x} to the root
- ▶ insert x as new root

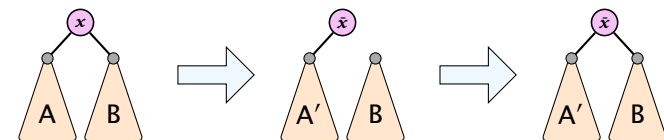


The illustration shows the case when \bar{x} is the predecessor of x .

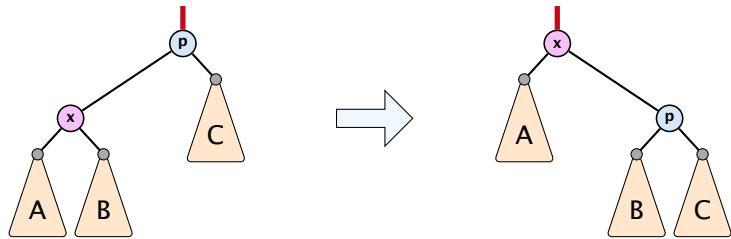
Splay Trees

delete(x)

- ▶ search for x ; $\text{splay}(x)$; remove x
- ▶ search largest element \bar{x} in A
- ▶ $\text{splay}(\bar{x})$ (on subtree A)
- ▶ connect root of B as right child of \bar{x}



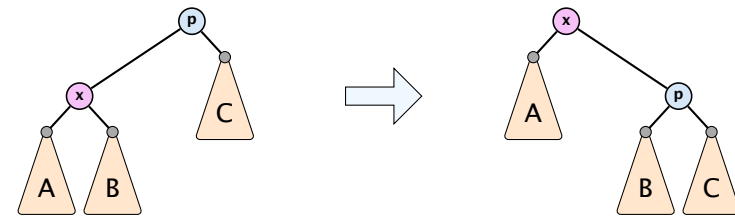
Move to Root



How to bring element to root?

- ▶ one (bad) option: $\text{moveToRoot}(x)$
- ▶ iteratively do rotation around parent of x until x is root
- ▶ if x is left child do right rotation otw. left rotation

Splay: Zig Case

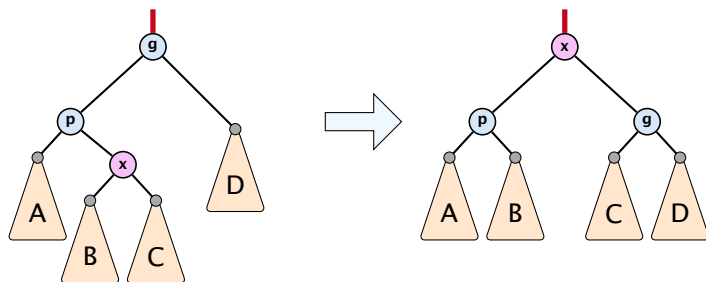


better option $\text{splay}(x)$:

- ▶ zig case: if x is child of root do left rotation or right rotation around parent

Note that $\text{moveToRoot}(x)$ does the same.

Splay: Zigzag Case

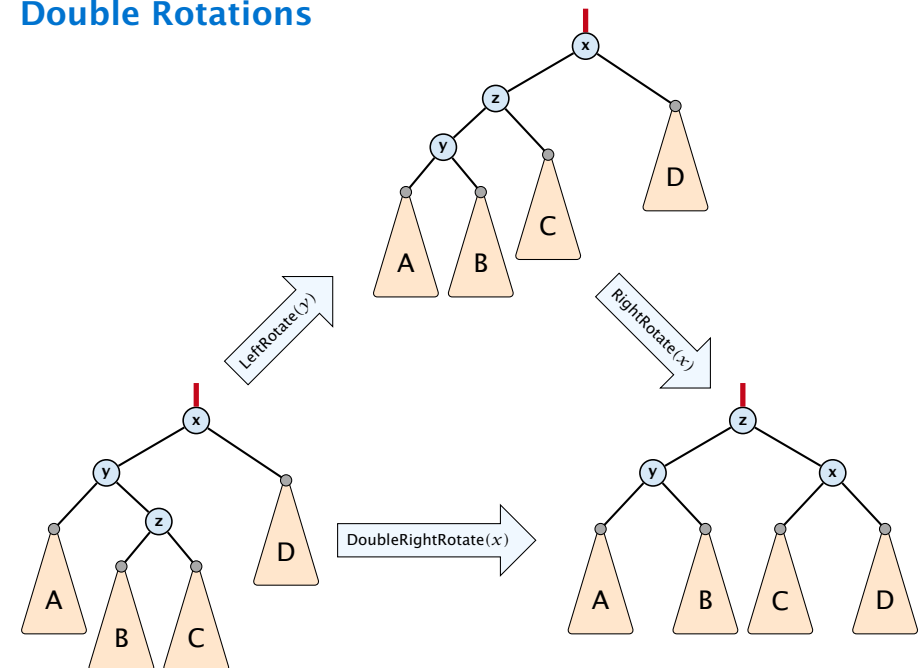


better option $\text{splay}(x)$:

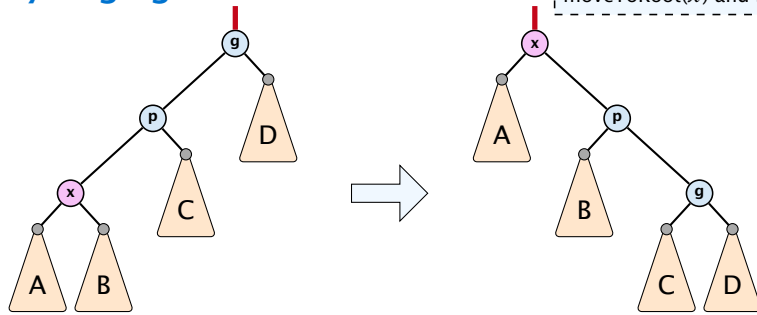
- ▶ zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

Note that $\text{moveToRoot}(x)$ does the same.

Double Rotations



Splay: Zigzig Case

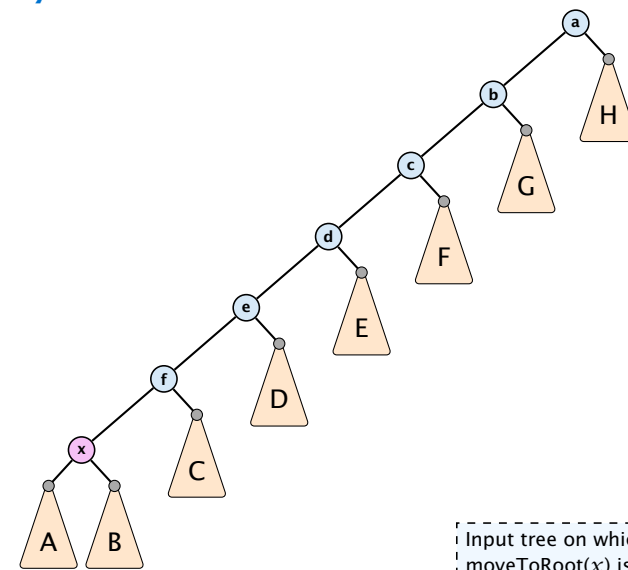


This case is different between $\text{moveToRoot}(x)$ and $\text{splay}(x)$.

better option $\text{splay}(x)$:

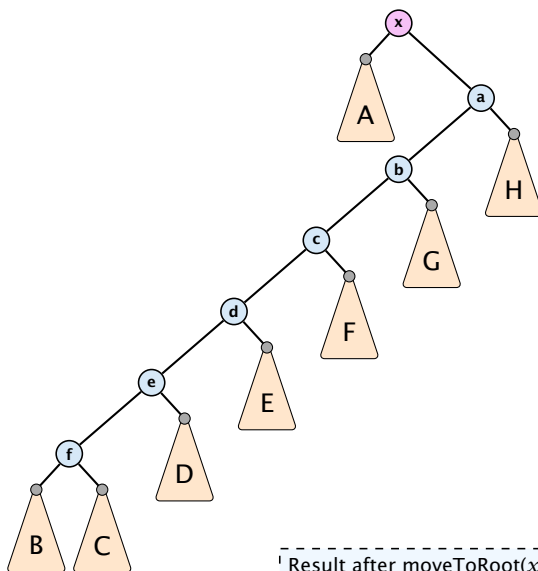
- ▶ zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- ▶ do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root



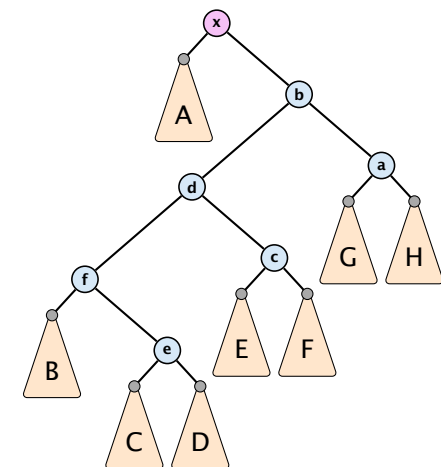
Input tree on which $\text{splay}(x)$ and $\text{moveToRoot}(x)$ is executed.

Splay vs. Move to Root



Result after $\text{moveToRoot}(x)$.

Splay vs. Move to Root



Result after $\text{splay}(x)$.

Static Optimality

Suppose we have a sequence of m find-operations. $\text{find}(x)$ appears h_x times in this sequence.

The cost of a **static** search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\min}))$, where T_{\min} is an **optimal static search tree**.

$\text{depth}_T(x)$ is the number of edges on a path from the root of T to x .
Theorem given without proof.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- ▶ the cost for accessing element x is $1 + \text{depth}(x)$;
- ▶ after accessing x the tree may be re-arranged through rotations;

Conjecture:

A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S .

Lemma 1

*Splay Trees have an **amortized** running time of $\mathcal{O}(\log n)$ for all operations.*

Amortized Analysis

Definition 2

A data structure with operations $\text{op}_1(), \dots, \text{op}_k()$ has amortized running times t_1, \dots, t_k for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most n elements, and let k_i denote the number of occurrences of $\text{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

- ▶ $\Phi(D_i)$ is the potential after the i -th operation.
- ▶ Amortized cost of the i -th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) .$$

- ▶ Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- ▶ **S.push()**
- ▶ **S.pop()**
- ▶ **S.multipop(k)**: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- ▶ **S.push()**: cost 1.
- ▶ **S.pop()**: cost 1.
- ▶ **S.multipop(k)**: cost $\min\{\text{size}, k\} = k$.

Example: Stack

Use potential function $\Phi(S) = \text{number of elements on the stack}$.

Amortized cost:

- ▶ **S.push()**: cost

$$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta\Phi = 1 + 1 \leq 2 .$$

- ▶ **S.pop()**: cost

$$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta\Phi = 1 - 1 \leq 0 .$$

- ▶ **S.multipop(k)**: cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0 .$$

Note that the analysis becomes wrong if pop() or multipop() are called on an empty stack.

Example: Binary Counter

Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n -bit binary counter may require to examine n -bits, and maybe change them.

Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
- ▶ Changing bit from 1 to 0: cost 1.
- ▶ Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g., 001101 has $k = 1$).

Example: Binary Counter

Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x .

Amortized cost:

- ▶ Changing bit from 0 to 1:

$$\hat{C}_{0 \rightarrow 1} = C_{0 \rightarrow 1} + \Delta\Phi = 1 + 1 \leq 2.$$

- ▶ Changing bit from 1 to 0:

$$\hat{C}_{1 \rightarrow 0} = C_{1 \rightarrow 0} + \Delta\Phi = 1 - 1 \leq 0.$$

- ▶ **Increment:** Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \rightarrow 0)$ -operations, and one $(0 \rightarrow 1)$ -operation.

Hence, the amortized cost is $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$.

Splay Trees

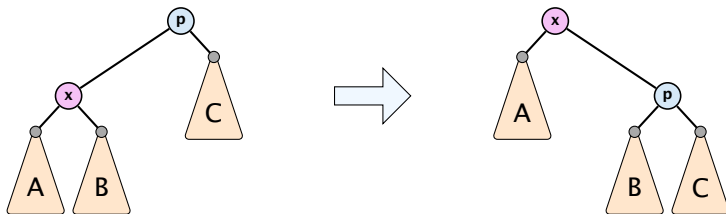
potential function for splay trees:

- ▶ size $s(x) = |T_x|$
- ▶ rank $r(x) = \log_2(s(x))$
- ▶ $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

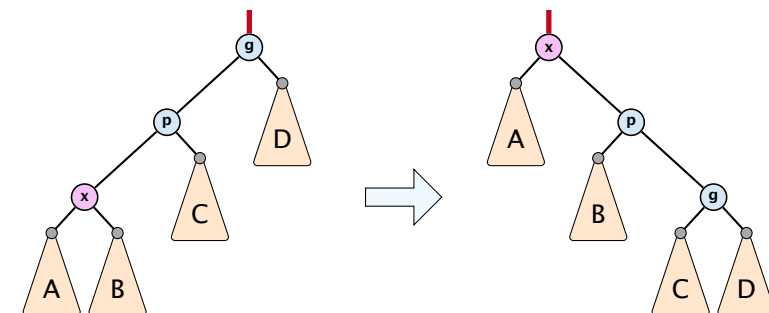
Splay: Zig Case



$$\begin{aligned} \Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x) \end{aligned}$$

$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

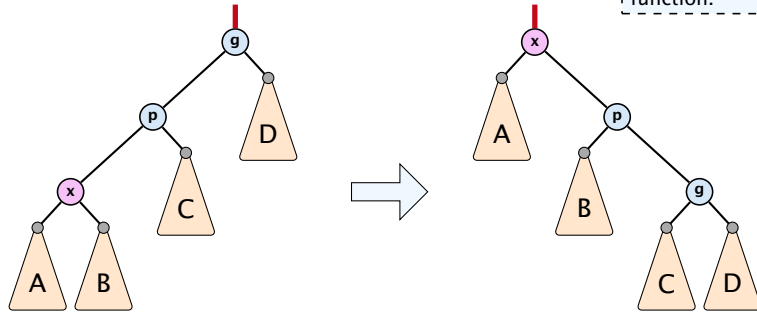
Splay: Zigzig Case



Last inequality follows from next slide.

$$\begin{aligned} \Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \\ &\leq -2 + 3(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzig}} \leq 3(r'(x) - r(x)) \end{aligned}$$

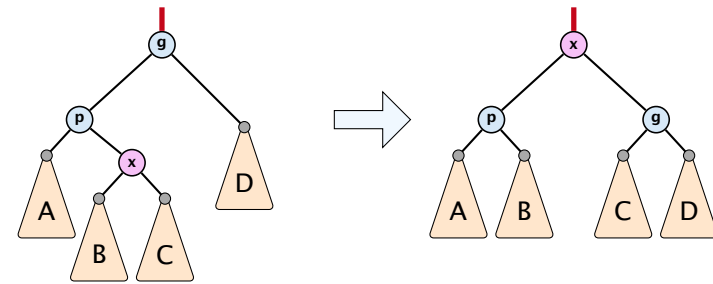
Splay: Zigzig Case



The last inequality holds because log is a concave function.

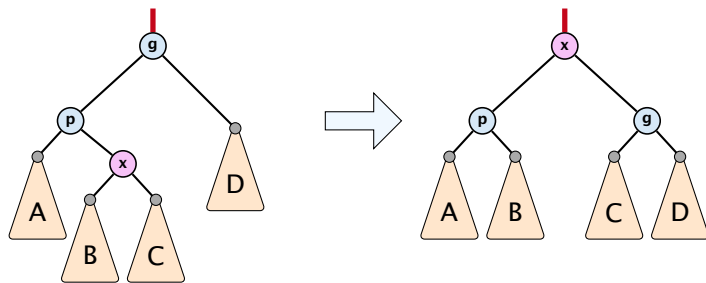
$$\begin{aligned} & \frac{1}{2}(r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2}(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \\ &= \frac{1}{2}\log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2}\log\left(\frac{s'(g)}{s'(x)}\right) \\ &\leq \log\left(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

Splay: Zigzag Case



$$\begin{aligned} \Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \end{aligned}$$

Splay: Zigzag Case



$$\begin{aligned} & \frac{1}{2}(r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2}(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))) \\ &\leq \log\left(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

Amortized cost of the whole splay operation:

$$\begin{aligned} & \leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + r(\text{root}) - r_0(x) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$

The first one is added due to the fact that so far for each step of a splay-operation we have only counted the number of rotations, but the cost is 1+#rotations.

The second one comes from the zig-operation. Note that we have at most one zig-operation during a splay.

Splay Trees

Bibliography

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